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The Small Dispersion Limit of the Korteweg-deVries Equations

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Contents

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Abstract

The scattering transform method is used to study the weak limit of solutions to the initial value problem for the Korteweg-deVries (KdV) equation as the dispersion tends to zero. In that limit the associated Schrödinger operator becomes semiclassical, so the exact scattering data is replaced by its corresponding WKB expressions. Only nonpositive initial data are considered; in that case the limiting reflection coefficient vanishes. The explicit solution of Kay and Moses for the reflectionless inverse transform is then analyzed, and the weak limit, valid for all time, is characterized by a quadratic minimum problem with constraints. With the aid of function theoretical methods, solutions of this minimum problem are constructed in terms of solutions to an initial value problem for some auxiliary functions.

The weak limit satisfies the KdV equation with the dispersive term dropped until the finite time at which its derivatives become infinite. Up to that time the weak limit is a strong L^2 limit. At later times the weak limit is locally described by Whitham's averaged equations or, more generally, by the equations found by Flaschka et al. using multiphase averaging. For large times, behavior of the weak limit is studied directly from the minimum problem.

0. The Zero Dispersion Limit for the KdV Equation

Introduction. In this study we analyze the behavior of solutions $u(x,t;\varepsilon)$ of

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0 \tag{1}$$

as $\varepsilon \to 0$ while the initial values are fixed:

$$u(x,0;\varepsilon) = u(x) \ . \tag{2}$$

It is known from computer studies that for t greater than a critical time, independent of ε, dependent only on the initial data, $u(x,t;\varepsilon)$ becomes oscillatory as $\varepsilon \to 0$. The wave length of these oscillations is $O(\varepsilon)$, their amplitude independent of ε. This shows that

$$\lim_{\varepsilon\to 0} u(x,t;\varepsilon) \tag{3}$$

exists only in the weak sense.

Our strategy for studying the limit (3) is similar to one employed by Hopf [5] and Cole [1] to study a similar but simpler singular perturbation, Burgers' equation:

$$u_t + uu_x = \varepsilon\, u_{xx} \ , \quad \varepsilon > 0 \ . \tag{4}$$

Hopf and Cole introduced a new variable V, defined by

$$U_x = u \ , \quad U = -\,2\varepsilon \log V \tag{5}$$

which linearizes equation (4):

$$V_t = \varepsilon V_{xx} \, .$$

Solutions of the linear heat equation can be explicitly expressed by an integral formula. This formula for V can be used to express u:

$$(6) \qquad u(x,t;\varepsilon) = \int \frac{x-y}{t} \exp\left(-\frac{D}{\varepsilon}\right) dy \Big/ \int \exp\left(-\frac{D}{\varepsilon}\right) dy$$

where

$$(7) \qquad D = U(y) + \frac{(x-y)^2}{2t}\,, \qquad U(y) = \int_{-\infty}^{y} u(z)\, dz\,,$$

and u(z) is the initial value. Letting $\varepsilon \to 0$ in (6) we get

$$(8) \qquad \bar{u}(x,t) = \lim_{\varepsilon \to 0} u(x,t;\varepsilon) = \frac{x-y^*}{t}$$

where $y^*(x,t)$ is that value of y which minimizes $D(x,y,t)$. The function y^* is defined for almost all x,t.

The key step in the method of Hopf and Cole sketched above is the transformation (5) that linearizes (4). Gardner, Green, Kruskal and Miura have discovered [3] that the KdV equation can be linearized by the so-called scattering transformation. In this study we use this linearization to trace the dependence of solutions on ε and to determine their limit as $\varepsilon \to 0$. The limit in this case is, for t large enough, only a weak limit. The initial functions we consider are assumed to tend to zero fast enough as $|x| \to \infty$, and to be nonpositive. The latter condition is used essentially in the analysis; it has been

shown in the dissertation of Venakides [13] how to remove this restriction.

The organization of the paper is as follows:

In Section 1 we solve the direct scattering problem for given initial data, assumed for simplicity to have a single local minimum. The direct problem is solved asymptotically; we then replace the exact initial data $u(x)$ by an approximate one $u(x;\varepsilon)$ whose exact scattering data are equal to the approximate data of $u(x)$. As shown subsequently, $\lim_{\varepsilon\to 0} u(x;\varepsilon) = u(x)$ in the $L_2(R)$ sense.

In Section 2 we use the Kay-Moses explicit solution of the reflectionless inverse problem, and carry out the limit $\varepsilon \to 0$. We show that $\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = \bar{u}(x,t)$ exists in the sense of weak convergence in $L_2(R)$ with respect to x, and that the weak limit $\bar{u}$ can be described as

(9) $$\bar{u} = \partial_{xx} Q^*.$$

The function $Q^*(x,t)$ is determined by solving a quadratic programing problem, i.e.

(10) $$Q^*(x,t) = \min_{0\leq\psi\leq\phi} Q(\psi;x,t) .$$

Here $Q(\psi;x,t)$ is a quadratic functional of ψ, which depends linearly on the parameters x,t. The function ϕ is determined by the initial data:

(11) $$\phi(\eta) = \mathrm{Re} \int \frac{\eta}{(-u(y)-\eta^2)^{1/2}}\, dy$$

We also show that $u^2(x,t;\varepsilon)$ leads, in an appropriate weak sense, to a limit.

In Section 3 we show that Q is continuous in a weak sequential topology, and that the space of admissible functions is compact in that topology. We further show that Q is a strictly convex function; since the admissible functions form a convex set, this implies not only that the minimum of Q is taken on at a unique function, but that this function is the only one which satisfies variational conditions.

The variational conditions are then converted to a Riemann-Hilbert problem, i.e. to the problem of determining an analytic function of class H^p in the upper half plane whose real and imaginary parts are prescribed on alternate intervals of the real axis.

In Section 4 we solve the Riemann-Hilbert problem for all times t that do not exceed t_b called the break-time, defined as the time beyond which the initial value problem

$$u_t - 6\,uu_x = 0\ ,\quad u(x,0) = u(x) \tag{12}$$

has no solution; equation (12) is obtained by setting $\varepsilon = 0$ in the KdV equation (1). We show that for $t < t_b$, $u(x,t;\varepsilon)$ tends to the solution of (10), and that this convergence is strong convergence in $L_2(R)$ with respect to x.

In Section 5 we solve the Riemann-Hilbert problem for any x and t and verify that the solution so obtained satisfies the variational conditions and therefore solves the minimum problem. The solution $\bar{u}$ determined from (9) has the form suggested by Whitham [14] and more generally by Flaschka, Forest and McLaughlin [2].

In Section 6 we determine the asymptotic behavior of $\bar{u}(x,t)$ for large t, For $0 < x/t < 4m^2$, where

$$m = -\min u(x) ,$$

we have

$$\bar{u}(x,t) \approx -\frac{1}{2\pi t}\,\phi(\sqrt{x/4t}) ;$$

here ϕ is the function defined in (11).

1. Asymptotic Analysis of the Direct Scattering Problem.

In a remarkable paper [3], Gardner, Greene, Kruskal and Miura (GGKM) have shown how to solve the initial value problem for the KdV equation

$$(1.1) \qquad u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0$$

for initial data u(x) that tends to zero so fast as $|x| \to \infty$ that

$$(1.2) \qquad \int (1 + x^2)\, |u(x)|\, dx < \infty .$$

Their method associates to each solution of KdV a one-parameter family of Schroedinger operators $\mathcal{L}(t)$:

$$(1.3) \qquad \mathcal{L}(t) = -\varepsilon^2 \partial_x^2 + u(x,t).$$

Recall that for a Schroedinger operator $\mathcal{L}$ whose potential u satisfies (1.2) one can define scattering data; these consist of

(i) The reflection coefficient R(k),

(ii) The eigenvalues $-\eta_n^2$, n = 1,...,N in the point spectrum of ($\eta_n > 0$ by convention),

(iii) The norming constants c_n associated with the eigenfunctions f_n.

We recall the definition of the norming constants: Let f be the eigenfunction of $\mathcal{L}$ with eigenvalue $-\eta^2$:

$$(1.4) \qquad \mathcal{L}f = -\varepsilon^2 f_{xx} + uf = -\eta^2 f,$$

normalized by

$$\int_{-\infty}^{\infty} f^2\, dx = 1. \tag{1.4)'$$

It is easy to show, using condition (1.2), that as $|x| \to \infty$, $f(x)$ decays exponentially. More precisely, there is a constant c such that

$$f(x) \simeq c\, e^{-\eta x/\varepsilon} \text{ as } x \to +\infty. \tag{1.5}$$

This constant c, chosen to be positive, is the norming constant.

The operation relating the potential u of to the scattering data $\{R(k),\eta_n,c_n\}$ is called the <u>scattering transform</u>. It can be inverted with the aid of the Gelfand-Levitan-Marchenko (GLM) equation [4]. GGKM have shown that if the potential u varies with a parameter t so that the KdV equation (1.1) is satisfied, then the scattering data vary in a particularly simple manner with t:

(1.6)
(i) $R(k,t) = R(k)\, e^{4ik^3 t/\varepsilon}$

(ii) η_n is independent of t

(iii) $c_n(t) = c_n\, e^{4\eta^3 t/\varepsilon}$

Formulas (1.6) have been derived from those given by GGKM, who treat the case $\varepsilon = 1$, by the simple rescaling

$$u(x,t) = v(x/\varepsilon,\ t/\varepsilon). \tag{1.7}$$

A similar rescaling has been applied at the end of this section to the formulas of Kay and Moses for the inverse scattering transform.

The GGKM solution of the initial value problem for KdV is to take

the scattering transform of the initial data, then apply (1.6) to determine the scattering data at time t, and then invert the scattering transform.

In this paper we shall study initial data which satisfy two further conditions in addition to (1.2):

$$u(x) \leq 0, \tag{1.8}$$

(1.8)′ $u(x)$ is C^1 and has only a finite number of critical points.

Condition (1.2) is there to make the machinery of scattering theory work smoothly. Formula (1.8) is a genuine restriction that allows us to neglect reflection. Condition (1.8)′ is there for technical reasons. In fact in this study we shall assume that $u(x)$ has a single critical point; we denote by x_0 its location. The changes necessary to handle the more general case (1.8)′ are not hard to make.

We study now the asymptotic behavior of the scattering data as $\varepsilon \to 0$. The distribution of the eigenvalues is governed by

<u>Theorem 1.1</u> (Weyl′s law): Let

$$\mathcal{L} = -\varepsilon^2 \partial_x + u$$

be a Schroedinger operator whose potential satisfies (1.2). Denote by $N(\varepsilon)$ the number of eigenvalues of $\mathcal{L}$; then

$$N(\varepsilon) \simeq \frac{\mathrm{Re}}{\pi\varepsilon} \int \sqrt{-u(y)}\; dy. \tag{1.9}$$

More generally, the number $N(\varepsilon,\eta)$ of eigenvalues less than $-\eta^2$ is

$$N(\varepsilon,\eta) \simeq \frac{\mathrm{Re}}{\pi\varepsilon} \int \sqrt{-u(y)-\eta^2}\, dy \tag{1.10}$$

Suppose u satisfies (1.8) and has a single minimum. Then for $-\eta^2 > \min u$ there are two functions $x_+(y)$ and $x_-(\eta)$, defined by

$$u(x_-) = u(x_+) = -\eta^2 , \quad x_- < x_+ \tag{1.11}$$

Clearly,

$$u(x) < -\eta^2 \quad \text{for} \quad x_- < x < x_+ ,$$

so that (1.10) can be written as

$$N(\varepsilon,\eta) \simeq \frac{1}{\pi\varepsilon} \Phi(\eta) \tag{1.12}$$

where

$$\Phi(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} \sqrt{-u(y)-\eta^2}\, dy , \tag{1.13}$$

see Figure 1:

We shall from now on assume that

(1.13)′ $$u(x_0) = \min u(x) = -1 .$$

Then the domain of the functions x_-, x_+ and Φ is $0 \leq \eta \leq 1$.

We turn next to the asymptotic determination of the norming constants. We shall use a crude WKB method, i.e. we represent the eigenfunction f in (1.4) as

(1.14) $$f(x) = c\, e^{-\theta(x)/\varepsilon} .$$

Substituting this into (1.4) gives

$$-\theta_x^2 + \varepsilon\, \theta_{xx} + u + \eta^2 = 0 .$$

We now set $\varepsilon = 0$ in this relation and define θ as the solution of

$$\theta_x = (\eta^2 + u)^{1/2} ,$$

which yields

(1.14)′ $$\theta(\eta,x) = \eta x + \int_x^\infty \eta - (\eta^2 + u(y))^{1/2}\, dy$$

We shall show below, see inequality (1.26), that

$$\int \sqrt{-u(y)}\, dy < \infty .$$

From this it follows easily that the integral (1.14) converges, and that as $x \to \infty$,

(1.15) $$\theta(\eta,x) = \eta x + O(1).$$

As x decreases from $+\infty$, θ is real and decreases until x reaches $x_+(\eta)$. We denote this minimum value of θ by

$$(1.16)\quad \theta_+(\eta) = \eta x_+(\eta) + \int_{x_+(\eta)}^{\infty} \eta - (\eta^2 + u(y))^{1/2}\, dy \,.$$

We now define

$$(1.17)\qquad c(\eta) = e^{\theta_+(\eta)/\varepsilon}$$

and define f to be given by formula (1.14), with θ and c defined by (1.14)′ and (1.17). We see that f satisfies the eigenvalue equation (1.4) crudely, and the normalization condition (1.4)′ just as crudely. It follows from (1.15) that f satisfies the asymptotic relation (1.5).

If we apply the WKB method, crude or refined, to calculate the reflection coefficient, we get $R(k) \equiv 0$.

Definition. We define the modified initial function $u(x;\varepsilon)$ as the function whose scattering data are:

(i) $R(k) \equiv 0$

(ii) η_n defined by

$$(1.18)\qquad \Phi(\eta_n) = (n - 1/2)\,\varepsilon\pi, \quad n = 1,\ldots,N(\varepsilon),$$

$$N(\varepsilon) = \left[\, \frac{1}{\varepsilon\pi}\,\Phi(0)\right]$$

(iii) $c_n = c(\eta_n), \quad n = 1,\ldots,N(\varepsilon),$

where $[\cdot]$ is the "greatest integer less than" function, $\Phi(\eta)$ is the function defined by (1.13) and $c(\eta)$ is the function defined by (1.17)

and (1.16). Definition $(1.18)_{ii}$ of η_n is the WKB result and is consistent with Theorem 1.1.

Theorem 1.2. The function $u(x;\varepsilon)$, defined above, approximates the prescribed initial data u in the following sense:

$$L^2 - \lim_{\varepsilon\to 0} u(x;\varepsilon) = u(x) \quad .$$

Proof of this theorem will be presented in Section 4.

Combining (1.18) and (1.6) gives the scattering data of $u(x,t;\varepsilon)$, the solution of KdV with initial data $u(x;\varepsilon)$:

$$c_n(t) = e^{(4\eta^3 t + \theta_+(\eta))/\varepsilon}, \quad \eta = \eta_n. \tag{1.19}$$

On account of $(1.18)_i$, the functions $u(x,t;\varepsilon)$ are reflectionless potentials. For these, Kay and Moses have given a simple solution of the GLM equation which leads to the following explicit expression of the potential u in terms of its scattering data:

$$u(x,t;\varepsilon) = \partial_x^2 W(x,t;\varepsilon) \tag{1.20}$$

where

$$W(x,t;\varepsilon) = -2\varepsilon^2 \log\det(I + G(x,t;\varepsilon)). \tag{1.20$'$}$$

Here G is the matrix

$$G = \varepsilon\left(\frac{e^{-(\eta_n x+\eta_m x)/\varepsilon}}{\eta_n + \eta_m} c_n c_m\right) \tag{1.21}$$

Using (1.19) we can rewrite this as

$$(1.21)' \qquad G = \varepsilon\left(\frac{g_n g_m}{\eta_n + \eta_m} \right) ,$$

where

$$(1.22) \qquad g_n(x,t;\varepsilon) = e^{-a(\eta_n,x,t)/\varepsilon} .$$

Here

$$(1.23) \quad a(\eta,x,t) = \eta x - 4\eta^3 t - \theta_+(\eta) ,$$

where $\theta_+(\eta)$ is defined in (1.16).

We define now some further auxiliary functions that will be of use in Section 2, and prove a few of their properties:

$$(1.24) \quad \theta_-(\eta) = \eta x_-(\eta) - \int_{-\infty}^{x_-(\eta)} \eta - \left(\eta^2 + u(y)\right)^{1/2} dy \ ;$$

we denote the derivative of Φ, defined in (1.13), by $-\phi$:

$$(1.25) \quad \phi(\eta) = -\frac{d}{d\eta}\Phi(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} \frac{\eta}{(-u(y)-\eta^2)^{1/2}}\, dy .$$

<u>Lemma 1.3</u>. ϕ is a nonnegative function which belongs to $L^1(0,1)$.

<u>Proof</u>: This follows from the fact that Φ is a decreasing function of η which is bounded. The first assertion is obvious; to verify the second we state

$$\Phi(0) = \int_{-\infty}^{\infty} (-u(y))^{1/2}\, dy = \int (-u(y))^{1/2}(1+y^2)^{1/2}(1+y^2)^{-1/2}\, dy$$

(1.26)

$$\leq \left[\int |u(y)|\,(1+y^2)\, dy \int \frac{1}{1+y^2}\, dy\right]^{1/2} < \infty$$

Here we used the Schwarz inequality, and assumption (1.2).

Lemma 1.4. The functions θ_+ and θ_-, defined by (1.16) and (1.24) respectively, are continuous and satisy

$$\theta_+(0) = \theta_-(0) = 0 ,$$

(1.27)

$$\theta_-(\eta) < \eta x_0 < \theta_+(\eta) \quad \text{for } 0 < \eta < 1 .$$

Proof: It follows directly from the definitions of θ_+ and θ_- that they are continuous for $\eta \neq 0$. To show contiuity, and (1.27), at $\eta = 0$ we argue as follows:

By assumption $|u(x)|$ is a decreasing function of x for x large enough. So for large x

$$(1.28) \qquad 2\int_{x/2}^{x} (-u(y))^{1/2} > x(-u(x))^{1/2} .$$

Since according to (1.26)

$$\int_{-\infty}^{\infty} \sqrt{-u(y)}\, dy < \infty ,$$

it follows from (1.28) that

(1.29) $$\lim_{x\to\infty} x\sqrt{-u(x)} = 0.$$

According to (1.11), $-u(x_+) = \eta^2$; from this and (1.29) we conclude that

(1.29)′ $$\lim_{\eta\to 0} x_+(\eta)\eta = 0.$$

This shows that the first term in the definition (1.16) of $\theta_+(\eta)$ tends to 0 as $\eta \to 0$.

We show now that the same is true of the second term. Introducing μ as new variable of integration by

$$u(y) = -\mu^2, \quad y = x_+(\mu),$$

we have, after integrating by parts

$$\int_{x_+(\eta)}^{\infty} \eta - (\eta^2+u(y))^{1/2}\, dy = -\int_0^{\eta} \eta-(\eta^2-\mu^2)^{1/2} \frac{dx_+(\mu)}{d\mu}\, d\mu$$

$$= \int_0^{\eta} \frac{\mu x_+(\mu)}{(\eta^2-\mu^2)^{1/2}}\, d\mu - \eta x_+(\eta) \leq \text{const.} \max_{\mu\leq\eta}\{\mu x_+(\mu)\}.$$

It follows from (1.29)′ that the above tends to 0 as $\eta \to 0$. This completes the proof of Lemma 1.4.

2. Asymptotic analysis of the inverse scattering problem.

In this section we carry out an asymptotic analysis as $\varepsilon \to 0$ of $u(x,t;\varepsilon)$, given by formula (1.20):

$$u(x,t;\varepsilon) = \partial_x^2 W(x,t;\varepsilon), \tag{2.1}$$

where W is defined by (1.20)′:

$$W(x,t;\varepsilon) = -2\varepsilon^2 \log \det(I + G(x,t;\varepsilon)) \quad , \tag{2.2}$$

and $G(x,t;\varepsilon)$ is given by formula (1.21).

We expand $\det(I + G)$ by grouping together all terms which only contain factors from G with indices from a set S. The resulting formula is

$$\det(I + G) = \sum_S \det G_S \; , \tag{2.3}$$

where G_S is the principal minor of G obtained by striking out all rows and columns whose index lies outside S. In the sum (2.3), S ranges over all subsets of the N indices and we take $\det G_S = 1$ when S is the null set.

We can factor G, given by (1.21)′, as

$$G = \varepsilon D\left(\frac{1}{\eta_n + \eta_m}\right) D \; ,$$

where D is the diagonal matrix with entries $g_1, \ldots, g_N$. Since the principal minors G_S of G have the same form as G, a similar expression holds for them. The matrix in the middle is of Cauchy type; its determinant is given by

$$\prod_{S\times' S} |\eta_n - \eta_m| \ / \ \prod_{S\times S} (\eta_n + \eta_m) ,$$

where $S\times' S$ means $S\times S$ minus the diagonal. Since $\det D = \prod_S g_n$ we conclude that

$$\det G_S = \left(\frac{\varepsilon}{2}\right)^{|S|} \prod_S \frac{g_n^2}{\eta_n} \prod_{S\times' S} \left|\frac{\eta_n - \eta_m}{\eta_n + \eta_m}\right| .$$

Using (1.22) we can write this as follows

(2.4) $$\det G_S = \left(\frac{\varepsilon}{2}\right)^{|S|} \exp\left(-Q_S/2\varepsilon^2\right) ,$$

where

(2.5) $$Q_S(x,t;\varepsilon) = 4\varepsilon \sum_S a(\eta_n,x,t) - \varepsilon^2 \sum_{S\times' S} \log\left(\frac{\eta_n - \eta_m}{\eta_n + \eta_m}\right)^2 + 2\,\varepsilon^2 \sum_S \log \eta_n .$$

Using (2.4) to express (2.3) gives

(2.6) $$\det(I + G) = \sum_{\text{all } S} \left(\frac{\varepsilon}{2}\right)^{|S|} \exp\left(-Q_S/2\varepsilon^2\right) .$$

We shall show that for small ε this sum is dominated by its largest term. Define

(2.7) $$Q^*(x,t;\varepsilon) = \min_S \left\{Q_S(x,t;\varepsilon)\right\} .$$

Since all terms in (2.6) are positive, and since there are $2^{N(\varepsilon)}$ of them, we get the following two sided inequality

$$\left(\frac{\varepsilon}{2}\right)^N e^{-Q^*/2\varepsilon^2} \leq \det(I + G) \leq 2^N e^{-Q^*/2\varepsilon}.$$

Taking the logarithm and multiplying by $2\varepsilon^2$ gives

$$2\varepsilon^2 N \log \left(\frac{\varepsilon}{2}\right) - Q^* \leq 2\varepsilon^2 \log \det (I + G) \leq 2\varepsilon^2 N \log 2 - Q^* .$$

The center term is just $-W$ as defined by (2.2) so we obtain the estimate

$$|W(x,t;\varepsilon) - Q^*(x,t;\varepsilon)| \leq 2\varepsilon^2 N(\varepsilon) \log \left(\frac{2}{\varepsilon}\right).$$

Since according to (1.9), $N(\varepsilon) = O(\varepsilon^{-1})$, we have proved

Theorem 2.1.

(2.8) $$\lim_{\varepsilon \to 0} [W(x,t;\varepsilon) - Q^*(x,t;\varepsilon)] = 0,$$
uniformly in x and t.

We shall show that the limit of $Q^*(x,t;\varepsilon)$ exists and can, just as each $Q^*(x,t;\varepsilon)$, be characterized as a solution of a minimum problem. To find this minimum problem we rewrite $Q_S(x,t;\varepsilon)$ given by (2.5) as a Stieltjes integral.

For each S we define $\psi_S(\eta,\varepsilon)$ as the distribution

(2.9) $$\psi_S(\eta,\varepsilon) = \varepsilon \pi \sum_S \delta(\eta - \eta_n)$$

and rewrite (2.5) as

$$(2.10)\quad Q_S(x,t;\varepsilon) = \frac{4}{\pi}\int_0^1 a(\eta,x,t)\,\psi_S(\eta,\varepsilon)\,d\eta - \frac{1}{\pi^2}\int_0^1\int_0^1 \log\left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \psi_S(\mu,\varepsilon)\times'\psi_S(\eta,\varepsilon)\,d\mu\,d\eta + \frac{2\varepsilon}{\pi}\int_0^1 \log\eta\,\psi_S(\eta,\varepsilon)\,d\eta\ ;$$

here $\psi_S(\mu,\varepsilon)\times'\psi_S(\eta,\varepsilon)$ is the product of the distributions minus the diagonal terms,

$$(2.11)\quad \psi_S(\mu,\varepsilon)\times'\psi_S(\eta,\varepsilon) = \varepsilon^2\pi^2 \sum_{S\times'S} \delta(\mu-\eta_m)\,\delta(\eta-\eta_n).$$

We define $\phi(\eta,\varepsilon)$ by

$$(2.12)\quad \phi(\eta,\varepsilon) = \varepsilon\pi\sum_{n=1}^{N(\varepsilon)} \delta(\eta-\eta_n)\ ;$$

Clearly

$$(2.13)\quad 0 \le \psi_S(\eta,\varepsilon)\,d\eta < \phi(\eta,\varepsilon)\,d\eta$$

as measures, for every S.

Since the η_n satisfy Weyl's law (Theorem 1.1), it follows that

$$(2.14)\quad \lim_{\varepsilon\to 0}\phi(\eta,\varepsilon)\,d\eta = \phi(\eta)\,d\eta,$$
$$\lim_{\varepsilon\to 0}\phi(\mu,\varepsilon)\times'\phi(\eta,\varepsilon)\,d\mu d\eta = \phi(\mu)\,\phi(\eta)\,d\mu\,d\eta,$$

in the sense of weak sequential convergence of measures, where $\phi(\eta)$ is defined by (1.25). These considerations suggest a minimum problem satisfied by the limit $Q^*(x,t;\varepsilon)$:

Theorem 2.2.

$$\lim_{\varepsilon \to 0} Q^*(x,t;\varepsilon) = Q^*(x,t) \tag{2.15}$$

uniformly on compact subsets of x,t, where

$$Q^*(x,t) = \min \{Q(\psi;x,t) : \psi \in A\}, \tag{2.16}$$

the admissible set A being all Lebesgue measurable functions ψ on [0,1] satisfying

$$0 \leq \psi(\eta) \leq \phi(\eta) \quad , \tag{2.17}$$

and $Q(\psi;x,t)$ denotes the quadratic form

$$Q(\psi;x,t) = \frac{4}{\pi} \int_0^1 a(\eta,x,t)\, \psi(\eta)\, d\eta - \frac{1}{\pi^2} \int_0^1 \int_0^1 \log \left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \psi(\mu)\, \psi(\eta)\, d\mu\, d\eta \ . \tag{2.18}$$

The proof is based on a series of lemmas and theorems.

Let A_ε denote the set of distributions $\psi_s(\eta,\varepsilon)$ defined by (2.9) for all possible sets of indices s and $\varepsilon > 0$,

$$A_\varepsilon = \{\psi_S(\eta,\varepsilon) : S,\ \varepsilon\} \ . \tag{2.19}$$

To the elements ψ of both the admissible sets A_ε and A we assign positive measures $\psi\, d\eta$ on [0,1]. If $\psi_S \in A_\varepsilon$ then from (2.13), (2.12), and $(1.18)_{ii}$

$$\int_0^1 \psi_S(\eta,\varepsilon)\, d\eta \leq \int_0^1 \phi(\eta,\varepsilon)\, d\eta = \varepsilon\, \pi\, N(\varepsilon) \leq \int_{-\infty}^{\infty} \sqrt{-u(y)}\, dy \ , \tag{2.20}$$

while if $\psi \in A$ then from (2.17) and (1.25)

$$\int_0^1 \psi(\eta)\, d\eta \le \int_0^1 \phi(\eta)\, d\eta = \int_{-\infty}^{\infty} \sqrt{-u(y)}\, dy \tag{2.21}$$

Thus we conclude from Lemma 1.3:

Lemma 2.3. The total variations of the measures $\psi\, d\eta$, ψ in A_ε and A are uniformly bounded.

Definition. A sequence of distributions ψ_k is said to w-converge to ψ if the corresponding measures $\psi_k(\eta)\, d\eta$ converge to $\psi(\eta) d\eta$ in the sense of weak sequential convergence of measures, that is if

$$\lim_{k\to\infty} (\chi,\psi_k) = (\chi,\psi) \tag{2.22}$$

for every continuous function χ. Here the parentheses denote the usual duality on [0,1]

$$(\chi,\psi) = \int_0^1 \chi(\eta)\, \psi(\eta)\, d\eta\ . \tag{2.23}$$

We shall abbreviate (2.22) as w-convergence and write

$$\operatorname*{w-lim}_{k\to\infty} \psi_k = \psi .$$

The next lemma characterizes the admissible set A as w-limits of the distributions in A_ε.

Lemma 2.4. (a) Let ψ_k denote a sequence of elements in A_ε , $\psi_k(\eta) = \psi_{S_k}(\eta,\varepsilon_k)$, such that as $\varepsilon_k \to 0$,

$$\operatorname*{w-lim}_{k\to\infty} \psi_k(\eta) = \psi(\eta); \tag{2.24}$$

then ψ belongs to A.

(b) Conversely, for each ψ in A we can choose $S(\varepsilon)$ so that

(2.25) $$\operatorname*{w-lim}_{\varepsilon\to 0} \psi_{S(\varepsilon)}(\eta,\varepsilon) = \psi(\eta).$$

(c)

(2.26) $$\operatorname*{w-lim}_{\varepsilon\to 0} \phi(\eta,\varepsilon) = \phi(\eta) .$$

Proof: By (2.13) the ψ_k satisfy

$$0 \le \psi_k(\eta)\, d\eta \le \phi(\eta,\varepsilon_k)\, d\eta .$$

Since inequalities between measures are preserved for their weak limits, it follows from (2.24), and (2.26) that

$$0 \le \psi(\eta)\, d\eta \le \phi(\eta)\, d\eta.$$

From this we conclude by the Radon-Nikodym theorem that ψ is a measurable function and that (2.17) is satisfied; thus $\psi \varepsilon A$.

Part (b) is straightforward and part (c) is contaied in (2.14).

The main step in proving Theorem 2.2 is in relating $Q_S(x,t;\varepsilon)$ as given in (2.10) to $Q(\psi;x,t)$ as given by (2.18).

<u>Theorem 2.5.</u> Denote by ψ_k a sequence of elements in A_ε , $\psi_k(\eta) = \psi_{S_k}(\eta,\varepsilon_k)$, such that as $\varepsilon_k \to 0$,

$$\operatorname*{w-lim}_{k\to\infty} \psi_k(\eta) = \psi(\eta) \quad ;$$

then

(2.27) $$\lim_{k\to 0} Q_{S_k}(x,t;\varepsilon_k) = Q(\psi;x,t)$$

for every x and t.

The mild technical difficulties in the proof arise in handling the second and third terms of the right hand side of (2.10). This will be accomplished with the aid of lemmas, and the introduction of the operator L:

$$(2.28) \qquad L\psi(\eta) = \frac{1}{2\pi} \int_0^1 \log \left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \psi(\mu)\, d\mu .$$

Lemma 2.6. $L\phi$ is a continuous function on $[0,\infty)$ that tends to zero as $\eta \to \infty$. In the interval $[0,1]$

$$(2.29) \qquad -L\phi(\eta) = \theta_+(\eta) - \theta_-(\eta) .$$

Proof: Since it was shown in Lemma 1.4 tha $\theta_\pm(\eta)$ are continuous functions, the continuity of $L\phi$ in $[0,1]$ follows from (2.29). The continuity of $L\phi$ on $[1,\infty)$ and behavior at infinity can be seen directly from formula (2.29).

Formula (2.29) will be derived at the end of Section 3.

Corollary 2.7.

(a) For every ψ in the admissible set A, $L\psi$ is continuous and

$$(2.29)' \qquad L\phi(\eta) \leq L\psi(\eta) \leq 0 \quad \text{for} \quad \eta \geq 0.$$

(b) $L\psi$ is an odd function of η.

Proof: (b) is obvious from (2.28) so we restrict our attention to $\eta \geq 0$. Inequality (2.29)' of (a) follows from the fact that by

inequality (2.17) admissible functions are nonnegative, and that the kernel of the integral operator L is negative.

To show the continuity of $L\psi$ at η, $\eta \geq 0$ we take any sequence $\{\eta_k\}$ converging to η with $\eta_k \geq 0$. The integrands of $L\psi(\eta_k)$ appearing in (2.28) converges almost everywhere to the integrand of $L\psi(\eta)$ and are bounded below by the integrands of $L\phi(\eta_k)$. Since, by the continuity of $L\phi$, we know $L\phi(\eta_k)$ converges to $L\phi(\eta)$, we conclude by a dominated convergence argument that $L\psi(\eta_k)$ converges to $L\psi(\eta)$. The continuity of $L\psi$ follows, completing the proof.

We use the operator L defined by (2.28) in conjunction with the duality notation of (2.23) to rewrite the functional $Q(\psi;x,t)$ of (2.18) as

(2.30) $$Q(\psi;x,t) = \frac{4}{\pi}\left[\left(a(\eta,x,t),\psi\right) - \frac{1}{2}(L\psi,\psi)\right] ;$$

note that by Lemma 1.4, $a(\eta,x,t)$ given by (1.23) is continuous. In particular this shows $Q(\psi,x,t)$ is finite for all ψ in the admissible set A.

Preparatory to proving Theorem 2.5, we break up the functionals $Q(\psi)$ and Q_S into two parts depending on a parameter B, which eventually tends to infinity:

(2.31) $$Q(\psi) = Q^B(\psi) + Q_B(\psi) ,$$

and

(2.32) $$Q_S = Q_S^B + Q_{B,S} .$$

Here, setting

$$\ell(\eta,\mu) \quad = \quad -\frac{1}{2\pi}\log\left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \quad ,$$

and

(2.33) $$\ell^B(\eta,\mu) \quad = \quad \min\{\ell(\eta,\mu),B\} \quad ,$$

we define

(2.34) $$Q^B(\psi) = \frac{4}{\pi}\int_0^1 a(\eta)\psi(\eta)d\eta + \frac{2}{2\pi}\int_0^1\int_0^1 \ell^B(\eta,\mu)\psi(\eta)\psi(\mu)\, d\eta\, d\mu \qquad ,$$

(2.35) $$Q_S^B \quad = \quad \frac{4}{\pi}\int_0^1 a(\eta)\,\psi_S(\eta)\,d\eta + \frac{2}{\pi}\int_0^1\int_0^1 \ell^B(\eta,\mu)\,\psi_S(\eta)\,\psi_S(\mu)\,d\eta\,d\mu .$$

<u>Lemma 2.8</u>. (a) If the sequence $\psi_k = \psi_{S_k}(\eta,\varepsilon_k)$ satisfies

$$\operatorname*{w-lim}_{k\to\infty} \psi_k(\eta) = \psi \ ,$$

then

$$\lim_{k\to\infty} Q^B_{S_k}(\varepsilon_k) = Q^B(\psi) \ .$$

(b) For all ψ in A

(2.36) $$|Q_B(\psi)| < \delta(B) \quad ,$$

where

(2.37) $$\lim_{B\to 0} \delta(B) = 0 \qquad .$$

(c) For all $\psi(\varepsilon)$ in A_ε

$$|Q_{S,B}| < \delta(B) + \gamma(\varepsilon) \qquad ,$$

$$|Q_{S,B}| < \delta(B) + \gamma(\varepsilon) \quad ,$$

where

$$\lim_{\varepsilon\to 0} \gamma(\varepsilon) = 0 \ .$$

<u>Proof</u>: Since both functions $a(\eta)$ and $\ell^B(\eta,\mu)$ are continuous, part (a) is a direct consquence of the definition of weak convergence (2.22).

To prove (2.31) and (2.34) we write

$$Q_B(\psi) = \frac{2}{\pi} \iint_{\ell > B} (\ell(\eta,\mu) - B)\ \psi(\eta)\ \psi(\mu)\ d\eta\ d\mu \ .$$

Since for admissible ψ,

$$0 \leq \psi \leq \phi,$$

it follows that

$$|Q_B(\psi)| \leq Q_B(\phi) \quad .$$

This proves (2.36) with $\delta(B) = Q_B(\phi)$; (2.37) follows from the observation that $Q(\phi) < \infty$.

For the proof of (c) we refer to [9]; we remark that it relies on the choice $(1.18)_{ii}$ of the η_n to show separately that

$$\varepsilon^2 \sum_{\ell > B}{}' \ell(\eta_n,\eta_m) - B < \delta(B) \ ,$$

and that

$$\varepsilon^2 \sum_1^N \log \eta_n < \gamma(\varepsilon)$$

In showing that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$ one has to use the fact that for a potential u that satisfies (1.2),

$$(2.38) \qquad \int (-u(x))^{1/2} \log (-u(x))\, dx < \infty .$$

Theorem 2.5 is an immediate consequence of Lemma 2.8.

The pointwise limit (2.69) of Theorem 2.5 is uniform on compact subsets of x,t space. This follows from

Lemma 2.9. The family $\{Q_S(x,t;\varepsilon): S,\varepsilon\}$ of linear functions of x and t defined in (2.10) is equicontinuous and equibounded on compact subsets of x,t space.

Proof: Since the functions $Q_S(x,t;\varepsilon)$ are linear, it suffices to show the coefficients of x, t and the quantity $Q_S(0,0;\varepsilon)$ are uniformly bounded. The coefficients of x and t are uniformly bounded since

$$\varepsilon \sum_S \eta_n \leq \varepsilon N \quad ,$$
$$\varepsilon \sum_S \eta_n^3 \leq \varepsilon N \quad ,$$

respectively and by (1.9) εN is bounded. A uniform bound for $Q_S(0,0;\varepsilon)$ will be demonstrated by considering the three terms of the right hand side of (2.10) separately. According to Lemma 1.4, $\theta_+(\eta)$ is continuous and therefore bounded, say $\theta_+(\theta) \leq M$. The first term of $Q_S(0,0;\varepsilon)$ is then uniformly bounded since

$$\varepsilon \sum_S \theta_+(\eta_n) \leq \varepsilon N M ,$$

and by (1.9) εN is bounded. The uniform bounds of the second and third terms of $Q_S(0,0;\varepsilon)$ follow directly from Lemma 2.8.

We now turn to Theorem 2.2.

Proof: Formula (2.15) of Theorem 2.2 will follow from two facts proved below:

(2.39) $$\limsup_{\varepsilon \to 0} Q^*(x,t;\varepsilon) \leq \inf \{Q(\psi;x,t) : \psi \in A\} ,$$

and

(2.40) $$\liminf_{\varepsilon \to 0} Q^*(x,t;\varepsilon) = Q(\psi^*;x,t) ,$$

for some ψ^* in the admissible set A. Comparing (2.39) and (2.40) we see that

(2.41) $$Q(\psi^*;x,t) = \inf \{Q(\psi;x,t) : \psi \in A\} .$$

Thus the minimum asserted in the definition (2.16) of $Q^*(x,t)$ is attained,

(2.42) $$Q^*(x,t) = Q(\psi^*;x,t) ,$$

and (2.15) follows.

To prove (2.39), let ψ be any element of the admissible set A. By Lemma 2.4, there exists $S(\varepsilon)$ such that

(2.43) $$\operatorname*{w-lim}_{\varepsilon \to 0} \psi_{S(\varepsilon)}(\eta,\varepsilon) = \psi(\eta) .$$

Let $\{\varepsilon_k\}$ be a sequence such that $\varepsilon_k \to 0$ and

$$(2.44)\qquad \lim_{k\to\infty} Q^*(x,t;\varepsilon_k) = \limsup_{\varepsilon \to 0} Q^*(x,t;\varepsilon)\ .$$

By definition (2.7) and Lemma 2.9, the $Q^*(x,t;\varepsilon)$ are the minima over subfamilies of an equibounded family of functions and so are themselves equibounded, thus insuring the limits of (2.44) are finite.

By (2.7)

$$Q^*(x,t;\varepsilon_k) \leq Q_{s_k}(x,t;\varepsilon_k)\ .$$

Using (2.43) and applying Theorem 2.5 we see that the right side above tends to $Q(\psi;x,t)$. Using (2.44) on the left we obtain in the limit

$$(2.45)\qquad \limsup_{\varepsilon \to 0}\ Q^*(x,t;\varepsilon) \leq Q(\psi;x,t)$$

for all $\psi \in A$. Since the left hand side of (2.45) is independent of ψ, we take the infimum of the right, obtaining (2.39).

To prove (2.40), let $S(\varepsilon)$ be such that the minimum of (2.7) is attained,

$$(2.46)\qquad Q_{S(\varepsilon)}(x,t;\varepsilon) = Q^*(x,t;\varepsilon)\ .$$

Let $\{\varepsilon_k\}$ be a sequence such that $\varepsilon_k \to 0$ and

$$(2.47)\qquad \lim_{k\to\infty} Q^*(x,t;\varepsilon_k) = \liminf_{\varepsilon \to 0} Q^*(x,t;\varepsilon).$$

Since by Lemma 2.3 the total variations of the measures $\psi_{S(\varepsilon_k)}(\eta,\varepsilon)\,d\eta$ are uniformly bounded, we apply the Helly selection theorem and extract a w-convergent subsequence. Passing to this subsequence we know by

Lemma 2.4 that the w-limit ψ^* lies in the admissible set A. Applying Theorem 2.5 to this sequence and using (2.46) we obtain

$$Q(\psi^*;x,t) = \lim_{k\to\infty} Q^*(x,t;\varepsilon_k), \tag{2.48}$$

which, along with (2.47), proves (2.40).

The limit in (2.15) is uniform on compact subsets of x and t since by definition (2.7) and Lemma 2.9, the $Q^*(x,t;\varepsilon)$ are minima over subfamilies of an equicontinuous family of functions and so are themselves equicontinuous. The result then follows from the Arzela-Ascoli theorem and the proof of Theorem 2.2 is complete.

Combining the limit (2.15) of Theorem 2.2 with (2.8) of Theorem 2.1 we conclude that

$$\lim_{\varepsilon\to 0} W(x,t;\varepsilon) = Q^*(x,t) \tag{2.49}$$

uniformly on compact subsets of x and t.

Let us denote convergence in the sense of distributions of functions of x and t by d-lim. Since derivatives of a uniformly convergent sequence of functions converge in the distribution sense we conclude from (2.1) and (2.49)

Theorem 2.10. Let $u(x,t;\varepsilon)$ be the solution of the KdV equation (1.5) with initial data $u(x;\varepsilon)$ given by (1.18); then

$$\operatorname*{d-lim}_{\varepsilon\to 0} u(x,t;\varepsilon) = \bar{u}(x,t) \tag{2.50}$$

exists, and

$$\bar{u}(x,t) = \partial_{xx} Q^*(x,t) \tag{2.51}$$

where $Q^*(x,t)$ is defined by (2.16).

Remark. Actually we have proved more; we have shown that $u(x,t;\varepsilon)$ tends to $\bar{u}(x,t)$ in the x-distribution sense. I.e., for any C_0^∞ function w of x,

$$\lim_{\varepsilon\to 0} (u(t;\varepsilon),w) = (\bar{u}(t),w) \; ; \tag{2.52}$$

furthermore the limit is uniform over compact subsets of t. We will further strengthen this result in Theorem 2.14.

Exploiting the concept of convergence in the distribution sense we shall now deduce as a corollary of Theorem 2.10 the following results:

Theorem 2.11. Let $u(x,t;\varepsilon)$ be as in Theorem 2.10, then the d-limit

$$\operatorname*{d-lim}_{\varepsilon\to 0} u^2(x,t;\varepsilon) = \overline{u^2}(x,t) \tag{2.53}$$

exists, and

$$\overline{u^2}(x,t) = \frac{1}{3}\,\partial_{xt}Q^*(x,t). \tag{2.54}$$

Similarly

$$\operatorname*{d-lim}_{\varepsilon\to 0} [u^3(x,t;\varepsilon) + \frac{3}{4}\,\varepsilon^2 u_x^2(x,t;\varepsilon)] = \overline{u^3}(x,t) \tag{2.55}$$

exists, and

$$\overline{u^3}(x,t) = \frac{1}{12}\,\partial_{tt}Q^*(x,t). \tag{2.56}$$

Proof: We rewrite the KdV equation (1.5) in conservation form

$$u_t - (3u^2)_x + \varepsilon^2 u_{xxx} = 0.$$

Substituting (2.1) into the first term and integrating with respect to x we see that $u(x,t;\varepsilon)$ satisfies

$$\partial_{xt}W - 3u^2 + \varepsilon^2 u_{xx} = 0\ .$$

Here we ascertain that the constant of integration is zero by letting $x \to \infty$. Solving for u^2 we obtain

$$u^2 = \frac{1}{3}\,\partial_{xt}W + \frac{\varepsilon^2}{3}\,u_{xx}\ . \tag{2.57}$$

The first term on the right hand side of (2.57) has by (2.49) the d-limit $\frac{1}{3}\,\partial_{xt}Q^*$ while by (2.50) the d-limit of the last term in (2.57) is zero. This shows that (2.53) and (2.54) hold.

To prove (2.55) and (2.56) we multiply the KdV equation (1.5) by 2u and write the result in conservation form:

$$(u^2)_t - (4u^3)_x + \varepsilon^2\big((u^2)_{xx} - 3u_x^2\big)_x = 0\ .$$

Substituting (2.57) into the first term and integrating with respect to x we see that $u(x,t;\varepsilon)$ satisfies

$$\frac{1}{3}\partial_{tt}W + \frac{\varepsilon^2}{3}u_{xt} - 4u^3 + \varepsilon^2(u^2)_{xx} - 3\varepsilon^2 u_x^2 = 0 .$$

Here, as before, we ascertain that the constant of integration is zero by letting $x \to \infty$. Solving for $u^3 + \frac{3}{4}\varepsilon^2 u_x^2$ we obtain

$$(2.58)\quad u^3 + \frac{3}{4}\varepsilon^2 u_x^2 = \frac{1}{12}\partial_{tt}W + \frac{\varepsilon^2}{12}u_{xt} + \frac{\varepsilon^2}{4}(u^2)_{xx}$$

The first term on the right hand side of (2.58) has by (2.49) the d-limit $\frac{1}{12}\partial_{tt}Q^*$, while the d-limits of the last two terms in (2.58) are by (2.50) and (2.53) zero. This shows that (2.55) and (2.56) hold, finishing the proof.

Theorem 2.10 can be strengthened:

Theorem 2.12. Let $u(x,t;\varepsilon)$ be as in Theorem 2.10. As functions of x, $u(\cdot,t;\varepsilon)$ converges to $\bar{u}(\cdot,t)$ weakly in $L^2(R)$ as $\varepsilon \to 0$ uniformly over compact subsets of t.

The proof requires

Lemma 2.13.

$$(2.59)\quad \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} u^2(x,t;\varepsilon)\,dx = \int_{-\infty}^{\infty} u^2(x)\,dx$$

uniformly in t.

Remark. It is not surprising that (2.59) is independent of t, since $\int u^2\,dx$ is conserved for solutions of KdV.

Proof: Making use of the explicit form of $W(x,t;\varepsilon)$, (2.2-2.5) and the fact that u_x vanishes as $|x| \to \infty$, we integrate (2.57):

$$(2.60)\quad \int_{\infty}^{\infty} u^2(x,t;\varepsilon)\,dx = \frac{1}{3}\,\partial_t W\Big|_{-\infty}^{\infty}$$

$$= -\frac{16}{3}\,\varepsilon\,\frac{\sum_{\text{all } S}(\Sigma_S\,\eta_n^3)\det G_S}{\Sigma_{\text{all } S}\det G_S}\Big|_{-\infty}^{\infty}$$

$$= \frac{16}{3}\,\varepsilon \sum_{n=1}^{N} \eta_n^3 .$$

Here we have taken limits of a weighted mean of $\Sigma_S\ \eta_n^3$ where the weights, $\det\ G_S$, are decreasing exponential functions of x, given by (2.4). At the upper limit the dominant term occurs when S is the null set while at the lower limit the dominant term occurs when S consists of all indices.

It follows from (2.26) of Lemma 2.4 and the definition (2.22) of w-convergence that

$$(2.61)\qquad \lim_{\varepsilon\to 0} \varepsilon\pi \sum_{n=1}^{N} \eta_n^3 = (\eta^3,\phi) .$$

Using the definition (1.25) of $\phi(\eta)$, interchanging the order of integration and performing the η integral gives

$$(\eta^3,\phi) = \int_0^1 \int_{x_-(\eta)}^{x_+(\eta)} \frac{\eta^4}{(-u(y)-\eta^2)^{1/2}}\,dy\,d\eta$$

$$(2.62)\qquad = \int_{-\infty}^{\infty} \int_0^{\sqrt{-u(y)}} \frac{\eta^4}{(-u(y)-\eta^2)^{1/2}}\,d\eta\,dy$$

$$= \frac{3\pi}{16} \int_{-\infty}^{\infty} u^2(y)\,dy \ .$$

Putting (2.60-2.62) together we obtain (2.59) and prove the lemma.

Proof of Theorem 2.12: By Lemma 2.13 we can find an $M > 0$ such that for all t and ε

$$(2.63) \qquad \| u(\cdot,t;\varepsilon)\|_{L^2} = \left(\int_{-\infty}^{\infty} u^2(x,t;\varepsilon)\, dx\right)^{1/2} \leq M .$$

Given any $v \in L^2(R)$ and $\delta > 0$, we can find a $C_0^\infty(R)$ function w such that

$$(2.64) \qquad \| v - w\|_{L^2} \leq \delta .$$

We write

$$(2.65) \qquad (u(t;\varepsilon),v) = (u(t;\varepsilon),w) + (u(t;\varepsilon),v-w) .$$

According to (2.52), the first term on the right converges, uniformly for bounded t, to

$$(\bar{u}(t),w) .$$

The second term is by the Schwarz inequality and the estimates (2.63) and (2.69) less than $M\delta$. It follows from this that the left side of (2.65) tends to a limit for every v. This proves that $u(x,t;\varepsilon)$ has a weak limit as $\varepsilon \to 0$, uniformly for bounded t. In view of (2.52), this weak limit can be identified with $\bar{u}$. This shows that $\bar{u}(x,t)$ is in L_2 with respect to x, and proves Theorem 2.12.

We end this section by remarking that one can show

$$(2.66) \qquad \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} u(x,t;\varepsilon)\, dx = \int_{-\infty}^{\infty} u(x)\, dx$$

uniformly in t using arguments similar to those that led to (2.59) of Lemma 2.13. Since $u(x,t;\varepsilon) \leq 0$, this fact can be used to replace the weak L^2 convergence in Theorem 2.12 by weak L^p convergence for $1 < p \leq 2$ just by using the obvious L^p analogs of the steps in the proof.

3. Analysis of the Minimum Problem.

Theorem 2.10 gives an explicit formula, (2.51), for $\bar{u}(x,t)$, the limit in the distribution sense of solutions to the KdV equation as the coefficient of dispersion tends to zero. The formula involves the function $Q^*(x,t)$ characterized by a minimum problem, (2.16), where the quantity $Q(\psi;x,t)$ to be minimized, (2.30), is a quadratic function of ψ and the functions ψ of the admissible set A are subject to two linear inequalities, (2.17), a so-called quadratic programming problem. In this section we begin to use this formula to study a variety of properties of the limit $\bar{u}(x,t)$ and the related limits $\overline{u^2}(x,t)$ and $\overline{u^3}(x,t)$ given by formulas (2.54) and (2.56) of Theorem 2.11.

We start with some easy observations:

Theorem 3.1. As a function of x and t, Q^* is

(a) ≤ 0

(b) continuous

(c) concave

(d) increasing in x

(e) decreasing in t.

Proof: Since $0 \in A$, it follows from (2.16) that $Q^*(x,t) \leq Q(0;x,t) = 0$; this proves (a). Using the definition (1.23) of $a(\eta,x,t)$ in the formula (2.30) for $Q(\psi;x,t)$ one sees

$$(3.1) \qquad Q(\psi;x,t) = \frac{4}{\pi}(\eta,\psi)x - \frac{16}{\pi}(\eta^3,\psi)t - \frac{4}{\pi}(\theta_+,\psi) - \frac{2}{\pi}(L\psi,\psi).$$

Since the admissible ψ satisfy $0 \leq \psi \leq \phi$ and ϕ belongs to $L^1[0,1]$, it follows easily from (3.1) that $\{Q(\psi;x,t)\colon \psi \in A\}$ is an equicontinuous

family of functions of x and t. It follows that Q^*, the infimum of an equicontinuous family of functions, is itself continuous.

Each function $Q(\psi;x,t)$, being linear in x, t, has properties (c), (d) and (e). It follows that so does Q^*, their infimum.

We draw now some conclusions from Theorem 3.1. From the concavity of Q^* we deduce that the matrix of second derivatives of Q^* is negative, in the distribution sense. Using expressions (2.51), (2.54) and (2.56) for these second derivatives yields

<u>Corollary 3.2.</u>

$$\xi^2\bar{u} + 6\xi\tau\overline{u^2} + 12\,\tau^2\overline{u^3} \leq 0, \tag{3.2}$$

for all real ξ, τ. In particular

$$\bar{u} \leq 0 \quad \text{and} \quad \overline{u^3} \leq 0. \tag{3.3}$$

We remark that there is no maximum principle like (3.3) for solutions of the KdV equation. That is, if the initial values of a solution $u(x,t)$ of the KdV equation satisfy $u(x,0) \leq 0$, one cannot conclude $u(x,t) \leq 0$. This may be seen by noting that if $u(x_0,0) = 0$, the value of the third x-derivative of u at x_0 could well be negative. It follows then from the KdV equation that $u_t(x_0,0) > 0$, and so $u(x_0,t) > 0$ for small $t > 0$. Assumption (1.8) is that $u(x) \leq 0$, thus once we show $\bar{u}(x,0) = u(x)$ then (3.3) can be interpreted as a kind of maximum principle for the zero dispersion limit of solutions to KdV.

To derive further information, some properties of the integral operator L, defined by (2.28) are needed. The first is a strengthening of Corollary 2.7 which asserted that for every ψ in the admissible set A, $L\psi(\eta)$ is a continuous function on R that vanishes at infinity.

Theorem 3.3. The set of functions $\{L\psi: \psi \in A\}$ is an equicontinuous subset of $C_0(R)$.

Proof: We break up the operator L into two parts depending on a parameter B which will tend to infinity:

$$L = L^B + L_B \quad , \tag{3.4}$$

where

$$L^B \psi(\eta) = -\int_0^1 \ell^B(\eta,\mu)\, \psi(\mu)\, d\mu \quad , \tag{3.5}$$

and ℓ^B is defied by (2.33).

As B tends to infinity, $L^B\phi(\eta)$ will tend to $L\phi(\eta)$, monotonically for all $\eta \in R$. Since these functions are in $C_0^\infty(R)$, it follows from Dini's theorem that the convergence is uniform in η. That is, if we define

$$\delta(B) = \max |L_B\phi| \quad , \tag{3.6}$$

then

$$\lim_{B\to 0} \delta(B) = 0 \; . \tag{3.7}$$

Let ψ be any admissible function; since admissibility means that

$$0 \le \psi(\eta) \le \phi(\eta) \; ,$$

and since the kernel of L_B is of one sign, it follows by (3.6) that for any η,

$$|L_B\psi| \le |L_B\phi| \le \delta(B) \; . \tag{3.8}$$

The kernel of L^B, defined in (2.33), is uniformly continuous. It follows that L^B maps any set bounded in the L^1 norm into an equicontinuous set. Since the admissible functions are L^1-bounded, it follows that they are mapped by L^B into an equicontinuous set. Since by (3.8), $L^B\psi$ differs from $L\psi$ at most by $\delta(B)$, and since $\delta(B) \to 0$ as $B \to \infty$, Theorem 3.3 follows.

Recalling the definition of w-convergence (2.22) for sequences in A, we use the previous result to prove the continuous dependence of L and Q on ψ:

Theorem 3.4. If ψ_k is a w-convergent sequence of elements in A:

$$\text{w-lim}\ \psi_k = \psi,$$

then

(3.9) $$\lim_{k\to\infty} L\psi_k(\eta) = L\psi(\eta) \quad ,$$

uniformly on R, and

(3.10) $$\lim_{k\to\infty} Q(\psi_k;x,t) = Q(\psi;x,t),$$

uniformly on compact subsets of x and t.

Proof: By Theorem 3.3, the functions $L\psi_k$ are equicontinuous, so by the Arzela-Ascoli theorem it is enough to show the convergence of (3.9) pointwise. Since the kernel of L^B is continuous, it follows from the definition of w-convergence that for each η ,

$$\lim L^B\psi_k(\eta) = L^B\psi(\eta)$$

Since by Theorem 3.3 $L^B\psi_k$ differs at most by $\delta(B)$ from $L\psi_k$, (3.9) follows.

Since the family of $Q(\psi;x,t)$ is equicontinuous in x and t, it suffices to show the pointwise convergence of (3.10). We consider separately the two terms on the right side of formula (2.30) for $Q(\psi;x,t)$. According to Lemma 1.4, $a(\eta,x,t)$ given by (1.22) is a continuous function of η and so by the definition (2.22) of w-convergence

$$\lim_{k\to\infty} (a,\psi_k) = (a,\psi) \tag{3.11}$$

for every x and t.

Next we consider the identity

$$(L\psi_k,\psi_k) - (L\psi,\psi) = (L\psi_k - L\psi,\ \psi_k) + (L\psi,\psi_k-\psi). \tag{3.12}$$

The first term on the right is bounded in absolute value by

$$\|L\psi_k - L\psi\|_{L^\infty}\|\psi_k\|_{L^1} \ .$$

Formula (3.9) shows that the first factor tends to zero; since ψ is admissible the second factor is bounded by $\|\phi\|_{L^1}$. Thus the first term on the right of (3.12) vanishes as $k \to \infty$. Corollary 2.7 asserts $L\psi$ is continuous so the second term on the right of (3.12) tends to zero as $k \to \infty$ by the w-convergence of ψ_k to ψ. This proves that

$$\lim_{k\to\infty} (L\psi_k,\psi_k) = (L\psi,\psi) . \tag{3.13}$$

Combining (3.11) and (3.13) according to formula (2.30) for $Q(\psi;x,t)$ we obtain (3.10) as asserted in the theorem.

An immediate consequence of (3.10) in Theorem 3.4 is

Corollary 3.5. The functional $Q(\psi;x,t)$ is a continuous function over $A\times R\times R$ where the admissible set A is given the weak sequential topology.

Proof: The result follows from (3.10) and the fact $Q(\psi;x,t)$ is jointly contiuous in x and t.

Theorem 3.6. The set of admissible functions A, defined by (2.17), is compact in the weak sequential topology; that is, every sequence in A contains a subsequence w-convergent to an element of A.

Proof: By Lemma 2.3, the total variations of the measures associated with A are uniformly bounded. It follows then from the Helly selection principle that every sequence of associated measures contains a subsequence $\{\psi_k(\eta)d\eta\}$ convergent to a measure $d\sigma$ in the sense

$$\lim_{k\to\infty} \int \chi(\eta)\, \psi_k(\eta)\, d\eta = \int \chi(\eta)\, d\sigma$$

for every continuous χ. We conclude from (2.17) that for all nonnegative χ

$$0 \leq \int \chi(\eta)\, d\sigma \leq \int \chi(\eta)\, \phi(\eta)\, d\eta .$$

This implies that the measure $d\sigma$ satisfies

$$0 \leq d\sigma \leq \phi(\eta)d\eta \quad ,$$

from which we conclude by the Radon-Nikodym theorem that $d\sigma = \psi(\eta)d\eta$ where ψ satisfies (2.17). Thus ψ_k is w-convergent to $\psi \in A$ and the proof of Theorem 3.6 is complete.

Combining the compactness of A asserted by Theorem 3.6 with the continuity of Q stated in Corollary 3.5, it will follow from a general topological principle that, for each x and t, $Q(\psi;x,t)$ takes on a minimum value over A. This gives a direct argument that the minimum problem (2.16) of Theorem 2.2 has a solution.

To prove uniqueness of the solution we need another property of the operator L.

Theorem 3.7. The integral operator is negative definite over all ψ in $L^1[0,1]$ for which $L\psi$ is in $L^\infty(R)$. This means

$$(3.14) \qquad -(L\psi,\psi) > 0$$

for all $\psi \neq 0$ in that class.

Proof: It is convenient to extend the functions ψ, originally defined on the interval [0,1], to all R by first setting $\psi(\eta) = 0$ for $\eta > 1$ and second, making ψ odd by setting $\psi(-\eta) = -\psi(\eta)$. This allows us to rewrite the operator L defined by (2.22), as a convolution operator,

$$(3.15) \qquad L\psi(\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log(\eta-\mu)^2 \, \psi(\mu) \, d\mu = - \ell * \psi(\eta) \; ,$$

where

(3.16) $$\ell(\eta) = -\frac{1}{2\pi}\log\eta^2.$$

Since $L\psi(\eta)$ is odd, we have

(3.17) $$-(L\psi,\psi) = \frac{1}{2}\langle \ell*\psi,\psi\rangle$$

where the brackets on the right denote the usual duality on R,

$$\langle\chi,\psi\rangle = \int_{-\infty}^{\infty} \chi(\eta)\,\psi(\eta)\,d\eta\ .$$

The definiteness of convolution operators is the subject of theorems going back to Carathéodory, Bochner and L. Schwartz. By (3.17), Theorem 3.7 follows from

Lemma 3.8. If $\psi \in L^1(R)$ is odd with compact support and $\ell * \psi \in L^\infty(R)$ then

(3.18) $$\langle \ell*\psi,\psi\rangle = \int_{-\infty}^{\infty} \frac{1}{|s|}\,|\hat{\psi}(s)|^2 ds\ ,$$

where $\hat{\psi}(s)$ is the Fourier transform given by

(3.19) $$\hat{\psi}(s) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{-i\eta s}\,\psi(\eta)\,d\eta.$$

Proof: Since ℓ is a tempered distribution, for all ψ in the Schwartz class S the Fourier transform of the convolution $\ell*\psi$ is $\sqrt{2\pi}\,\hat{\ell}\hat{\psi}$. By the Parseval relation

(3.20) $$\langle \ell*\psi,\psi\rangle = \sqrt{2\pi}\,\langle\hat{\ell}\hat{\psi},\bar{\hat{\psi}}\rangle = \sqrt{2\pi}\,\langle\hat{\ell},|\hat{\psi}|^2\rangle\ .$$

The Fourier transform of ℓ in the sense of the theory of distributions is

$$\hat{\ell}(s) = \frac{1}{(2\pi)^{1/2}} \text{ F.P. } \frac{1}{|s|}$$

where the F.P. indicates the integral is a Hadamard finite part.

In particular if $\chi(0) = 0$ then

$$(3.21) \qquad \langle \hat{\ell}, \chi \rangle = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \frac{1}{|s|} \chi(s) \, ds \ .$$

We apply this now to $\chi(s) = |\hat{\psi}(s)|^2$; taking ψ to be an odd function, (3.19) shows

$$\hat{\psi}(0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \psi(\eta) \, d\eta = 0$$

and allows us to employ (3.21) in conjunction with (3.20) to obtain

$$(3.22) \qquad \langle \ell * \psi, \psi \rangle = \int_{-\infty}^{\infty} \frac{1}{|s|} |\hat{\psi}(s)|^2 ds$$

for all odd ψ in S.

Now let ψ be any odd compactly supported member of $L^1(R)$ for which

$\ell * \psi$ is in $L^\infty(R)$. We approximate ψ by a sequence ψ_n of functions in S defined as follows:

$$(3.23) \qquad \psi_n = j_n * \psi \quad ,$$

where the mollifier j_n is given by

$$(3.24) \qquad \begin{aligned} j_n(\eta) &= n\, j(n\eta) \quad , \\ j(\eta) &= \frac{1}{(2\pi)^{1/2}} \exp\left(-\eta^2/2\right) . \end{aligned}$$

Since ψ is odd with compact support and the j_n are even elements of S, each ψ_n is an odd member of S and therefore satisfies (3.22). By classical properties of mollifiers we know that since ψ is in $L^1(R)$

$$(3.25) \qquad \lim_{n\to\infty} \|\psi_n - \psi\|_{L^1} = 0 ,$$

and since $\ell * \psi$ is in $L^\infty(R)$

$$(3.26) \qquad \|\ell * \psi_n\|_{L^\infty} = \|j_n * \ell * \psi\|_{L^\infty} \leq \|\ell * \psi\|_{L^\infty} .$$

Using (3.26) and the fact ℓ is even, see (3.16), we get the estimate

$$\begin{aligned} |\langle \ell * \psi_n, \psi_n\rangle - \langle \ell * \psi, \psi\rangle| &= |\langle \ell * \psi_n + \ell * \psi, \psi_n - \psi\rangle| \\ &\leq 2\|\ell * \psi\|_{L^\infty} \|\psi_n - \psi\|_{L^1} \quad , \end{aligned}$$

which by (3.25) implies

$$\lim_{n\to\infty} \langle \ell * \psi_n, \psi_n \rangle = \langle \ell * \psi, \psi \rangle. \tag{3.27}$$

Since (3.22) is satisfied for ψ_n we deduce from (3.27) that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{1}{|s|} |\hat{\psi}_n(s)|^2 ds = \langle \ell * \psi, \psi \rangle . \tag{3.28}$$

However from (3.24) we know

$$\hat{j}_n(s) = \frac{1}{(2\pi)^{1/2}} \exp\left(-s^2/2n^2\right) ,$$

which along with (3.23) gives

$$\begin{aligned}\hat{\psi}_n(s) &= \sqrt{2\pi}\, \hat{j}_n(s)\, \hat{\psi}(s) \\ &= \exp\left(-s^2/2n^2\right) \hat{\psi}(s) ,\end{aligned}$$

so the integral on the left hand side of (3.28) converges to integral in (3.18), completing the proof of both the lemma and Theorem 3.7.

An important consequence of the negative definiteness of L is:

Theorem 3.9. For each given x and t, the functional $Q(\psi;x,t)$ assumes its minimum at exactly one element, denoted $\psi^*(x,t)$, of the admissible set A.

Proof: Every ψ in A satisfies $0 \leq \psi \leq \phi$ and ϕ is in $L^1[0,1]$ so we know ψ is also in $L^1[0,1]$. By Corollary 2.7, $L\psi$ is an element of $C_0(R)$ and therefore also $L^\infty(R)$. We can then apply Theorem 3.7 to the difference of functions in A to prove that $Q(\psi;x,t)$ is strictly convex

over the set A. Since A is a convex set, $Q(\psi)$ assumes its minimum at exactly one admissible ψ.

Theorem 3.10. (a) $\psi^*(x,t)$ depends continuously on x, t in the sense of w-convergence.

(b) $L\psi^*(x,t)$ depends continuously on x,t in the maximum norm.

Proof: Suppose $(x_n,t_n) \to (x,t)$. Since, according to Theorem 3.6, A is compact, the sequence $\psi^*(x_n,t_n)$ has a cluster point ψ_* in A; a subsequence of $\psi^*(x_n,t_n)$ w-converges to ψ_*. Since by Corollary 3.5, $Q(\psi;x,t)$ is a continuous function of ψ, and since by definition $Q^*(x_n,t_n) = Q(\psi^*(x_n,t_n),x_n,t_n)$, we conclude that over this subsequence

$$(3.29) \qquad \lim_{n\to\infty} Q^*(x_n,t_n) = \lim_{n\to\infty} Q(\psi^*(x_n,t_n);x_n,t_n) = Q(\psi_*;x,t).$$

On the other hand, since by part (b) of Theorem 3.1, Q^* is a continuous function of x and t

$$\lim_{n\to\infty} Q^*(x_n,t_n) = Q^*(x,t).$$

Comparing this with (3.29) we see $Q^*(x,t) = Q(\psi_*;x,t)$ which means that ψ_* solves the minimum problem at (x,t) and so, by the uniqueness of Theorem 3.9, $\psi_* = \psi^*(x,t)$. Since $\psi^*(x,t)$ is therefore the only cluster point of $\psi^*(x_n,t_n)$ and A is compact, we conclude

$$\text{w-}\lim_{n\to\infty} \psi^*(x_n,t_n) = \psi^*(x,t) \ ;$$

this proves the continuity of ψ^*. The continuity of $L\psi^*$ follows since, by Theorem 3.4, L is sequentially continuous from A into $C_0(R)$ and the proof is complete.

The argument given in Theorem 3.1 for the continuity of $Q^*(x,t)$ shows that Q^* is even Lipschitz continuous in x and t. But now we can show more is true.

Theorem 3.11. $Q^*(x,t)$ is a continuously differentiable function of x and t with

$$\partial_x Q^*(x,t) = \frac{4}{\pi}(\eta,\psi^*(x,t)) ,$$

(3.30)

$$\partial_t Q^*(x,t) = -\frac{16}{\pi}(\eta^3,\psi^*(x,t)).$$

Proof: We deduce from formula (3.1) that

$$(3.31) \quad Q(\psi;x,t) - Q(\psi;y,s) = \frac{4}{\pi}(\eta,\psi)(x-y) - \frac{16}{\pi}(\eta^3,\psi)(t-s)$$

for all $\psi \varepsilon A$. Since ψ^* minimizes Q, we have the two-sided inequality

$$Q(\psi^*(x,t);x,t)-Q(\psi^*(x,t);y,s) \leq Q^*(x,t) - Q^*(y,s)$$
$$\leq Q(\psi^*(y,s);x,t) - Q(\psi^*(y,s);y,s) ,$$

which when combined with (3.31) gives

$$\frac{4}{\pi}(\eta,\psi^*(x,t))(x-y) - \frac{16}{\pi}(\eta^3,\psi^*(x,t))(t-s) \leq Q^*(x,t) - Q^*(y,s)$$
$$\leq \frac{4}{\pi}(\eta,\psi^*(y,s))(x-y) - \frac{16}{\pi}(\eta^3,\psi^*(y,s))(t-s).$$

Applying the continuity of ψ^* in the w-topology proved in Theorem 3.10 to both sides of the above two-sided inequality yields the formulas (3.30) and shows Q^* is continuously differentiable, proving the theorem.

The solution of a minimum problem satisfies variational conditions. For a minimum problem with constraints, such as the one we are dealing with, the variational conditions take the form (suppressing x and t)

$$(3.32) \qquad \frac{d}{d\varepsilon} Q(\psi^* + \varepsilon\chi)\big|_{\varepsilon=0} \geq 0$$

for all functions χ such that $\psi^*+\chi$ is admissible. Since

$$Q(\psi+\varepsilon\chi) = Q(\psi) + \varepsilon \frac{4}{\pi} (a-L\psi,\chi) - \varepsilon^2 \frac{2}{\pi} (L\chi,\chi),$$

we can restate (3.22) so

$$(3.32)' \qquad (a - L\psi^*,\chi) \geq 0.$$

<u>Definition</u>. We say that ψ in A satisfies the variational condition for minimizing $Q(\psi)$ in A if

$$(3.33) \qquad \frac{d}{d\varepsilon} Q(\psi + \varepsilon\chi)\big|_{\varepsilon=0} = (a - L\psi,\chi) \geq 0$$

for all χ such that

$$(3.33)' \qquad \psi + \chi \in A .$$

Suppose ψ satisfies (3.33); set

$$\chi_+ = \begin{cases} -\psi & \text{where } a-L\psi > 0 \\ 0 & \text{otherwise}, \end{cases}$$

$$\chi_- = \begin{cases} \phi-\psi & \text{where } a-L\psi < 0 \\ 0 & \text{otherwise}. \end{cases}$$

Clearly, χ_+ and χ_- satisfy (3.33)'. By construction, $(a - L\psi, \chi_\pm) \leq 0$; this is the opposite of (3.33), which implies that

$$(a-L\psi, \chi_\pm) = 0.$$

This implies that

$$\psi = \begin{cases} 0 & \text{where } a-L\psi > 0 \\ \phi & \text{where } a-L\psi < 0. \end{cases} \tag{3.34}$$

It is easy to verify that, conversely, if ψ satisfies (3.34), then (3.33) holds for all ψ, satisfying (3.33)'.

<u>Theorem 3.12</u>. If ψ in A satisfies the variational condition, then $\psi = \psi^*$.

Proof: Since $\chi = \psi^*-\psi$ satisfies (3.33)′, ψ satisfies (3.33) with $\chi = \psi^*-\psi$,

$$(a - L\psi,\ \psi^* - \psi) \geq 0 .$$

Since the minimizing ψ^* also satisfies the variational condition, and $\chi = \psi-\psi^*$ satisfies (3.33)′, it satisfies (3.32)′:

$$(a-L\psi^*,\psi-\psi^*) \geq 0 .$$

We add these inequalities and obtain

$$(L(\psi^*-\psi),\psi^*-\psi) \geq 0.$$

Using the negative definiteness of L we conclude that $\psi^*-\psi = 0$.

The importance of Theorem 3.12 lies in this: in order to show that ψ^* solves the minimum problem (2.16) it suffices to verify that it satisfies the variational conditions (3.34). We will exploit this idea in Sections 4 and 5.

An easy direct consequence of the variational conditions (3.34) that we will need in later sections is

<u>Theorem 3.13</u>. For all x and t

$$\psi^*(x,t) = \begin{cases} 0 & \text{where } \eta x-4\eta^3 t - \theta_+(\eta) > 0 \\ \\ \phi & \text{where } \eta x-4\eta^3 t - \theta_-(\eta) < 0. \end{cases} \tag{3.35}$$

Proof: By (2.42) of Corollary 2.7

$$a \leq a-L\psi^* \leq a-L\phi .$$

Since by Lemma 2.6, $-L\phi(\eta) = \theta_+(\eta) - \theta_-(\eta)$ and by definition (1.22), $a(\eta,x,t) = \eta x-4\eta^3 t - \theta_+(\eta)$, we see the inequality can be rewritten as

$$\eta x-4\eta^3 t - \theta_+(\eta) \leq a-L\psi^* \leq \eta x-4\eta^3 t - \theta_-(\eta).$$

Clearly (3.35) follows from this inequality and the variational conditions (3.34); and the theorem is proved.

Motivated by the variational conditions, we partition $[0,1]\times R\times R$ into three disjoint subsets:

$$\begin{aligned} I^0 &= \{(\eta,x,t): a - L\psi^* = 0\} \\ I^+ &= \{(\eta,x,t): a - L\psi^* > 0\} \qquad (3.36) \\ I^- &= \{(\eta,x,t): a - L\psi^* < 0\}. \end{aligned}$$

From Theorem 3.10 we see that $a-L\psi^*$ is continuous on $[0,1]\times R\times R$ and so the sets I^+ and I^- are open while I^0 is closed and separates I^+ and I^-. We denote the (x,t) slices of I^0, I^+, and I^- by $I^0(x,t)$, $I^+(x,t)$ and $I^-(x,t)$ respectively where, for instance

(3.37) $$I^0(x,t) = \{\eta: (\eta,x,t) \in I^0\}.$$

Given the sets $I^0(x,t)$, $I^+(x,t)$ and $I^-(x,t)$, one can determine $\psi^*(x,t)$ by

Theorem 3.14. If $\psi \in L^1[0,1]$ and $L\psi \in L^\infty(R)$ such that ψ satisfies

(3.38) $$L\psi = a \quad \text{on} \quad I^0(x,t) \quad ,$$

and

(3.39) $$\begin{aligned} \psi &= 0 \text{ on } I^+(x,t) \\ \psi &= \phi \text{ on } I^-(x,t) \end{aligned}$$

then $\psi = \psi^*(x,t)$.

Proof: That $\psi^*(x,t)$ satisfies (3.38) and (3.39) is clear from the construction (3.36) of the sets I^0, I^+, and I^-, and from the variational conditions (3.34). Now if ψ satisfies (3.38) and (3.39) then the difference $\psi_0 = \psi - \psi^*(x,t)$ satisfies the homogeneous version of (3.38) and (3.39):

(3.40) $$\begin{aligned} L\psi_0 &= 0 \text{ on } I^0(x,t) \ , \\ \psi \quad &= 0 \text{ off } I^0(x,t). \end{aligned}$$

This implies that the product $\psi_0 \cdot L\psi_0 = 0$ for all η, so that

$$(L\psi_0, \psi_0) = 0.$$

But, according to Theorem 3.7, L is negative definite so we conclude $\psi_0 = 0$, proving the theorem.

We now establish the last major property of the operator L. We

extend the functions ψ defined on $[0,1]$ to all of R by setting $\psi(\eta) = 0$ for $\eta > 1$ and taking ψ to be odd: $\psi(-\eta) = -\psi(\eta)$. The set of all functions obtained by extending elements of $L^p[0,1]$ $(p \geq 1)$ in this fashion we denote $L^p_0(R)$. Clearly $L^p_0(R)$ is just the set of all odd $L^p(R)$ functions supported in $[-1,1]$.

Theorem 3.15. If $\psi \in L^p_0(R)$ for some p, $1 < p < \infty$, then $L\psi \in C_0(R)$ and

$$L\psi(\eta) = \int_0^\eta H\psi(\tau)\, d\tau \ , \tag{3.41}$$

where H is the Hilbert transform:

$$H\psi(\eta) = \frac{\text{P.V.}}{\pi} \int_{-\infty}^{\infty} \frac{1}{\eta-\mu}\, \psi(\mu)\, d\mu . \tag{3.42}$$

Proof: Since the Hilbert transform takes L^p functions into L^p functions for $1 < p < \infty$ such that

$$H\psi(\tau) = L^p\text{-}\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau-\mu}{(\tau-\mu)^2+\varepsilon^2}\, \psi(\mu)\, d\mu ,$$

we conclude

$$\int_0^\eta H\psi(\tau)\, d\tau = \lim_{\varepsilon \to 0} \int_0^\eta \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau-\mu}{(\tau-\mu)^2+\varepsilon^2}\, \psi(\mu)\, d\mu\, d\tau .$$

The order of integration can then be exchanged by Fubini's theorem and the τ integration can be carried out to obtain

$$\int_0^\eta H\psi(\tau)d\tau = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{1}{2\pi} \log \frac{(\eta-\mu)^2+\varepsilon^2}{\mu^2+\varepsilon^2} \psi(\mu)\, d\mu$$

(3.43)

$$= \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_0^{\infty} \log \frac{(\eta-\mu)^2+\varepsilon^2}{(\eta+\mu)^2+\varepsilon^2} \psi(\mu)\, d\mu \;.$$

Here the oddness of ψ was used in the last step. Since for fixed η

$$L^q - \lim_{\varepsilon\to 0} \frac{1}{2\pi} \log \frac{(\eta-\mu)^2+\varepsilon^2}{(\eta+\mu)^2+\varepsilon^2} = \frac{1}{2\pi} \log \left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \quad ,$$

where $1/p + 1/q = 1$, we see from (3.43) that

$$(3.44) \quad \int_0^\eta H\psi(\tau)\, d\tau = \frac{1}{2\pi} \int_0^1 \log \left(\frac{\eta-\mu}{\eta+\mu}\right)^2 \psi(\mu)\, d\mu \quad ,$$

which by the definition of L, (2.28), proves (3.41).

The continuity of $L\psi$ follows directly from formula (3.41). It can be seen from the definition of L that the kernel $\frac{1}{2\pi} \log \left(\frac{\eta-\mu}{\eta+\mu}\right)^2$ tends to zero uniformly over the support of ψ as $|\eta|$ tends to infinity. Thus $L\psi$ vanishes at infinity and the theorem is established.

Recall now the relation of the Hilbert transform to analytic functions: Let H^p $(1 \le p < \infty)$ denote the space of all functions $f(\zeta)$ satisfying

(1) $f(\zeta)$ is analytic in the upper half plane $Im(\zeta) > 0$.

(2) There exists a constant $C > 0$ such that

$$\left(\int_{-\infty}^{\infty} |f(\eta+i\tau)|^p\, d\eta \right)^{1/p} < C, \text{ for all } \tau > 0.$$

Such functions have boundary values in $L^p(R)$ in the sense that $L^p\text{-}\lim_{\tau \to 0} f(\eta + i\tau) = f(\eta)$ exists.

The following result is classical [7]:

Theorem 3.16. If $\psi \varepsilon L^p$ for some p, $1 < p < \infty$, and ψ is real, then ψ and $H\psi$ are the real and imaginary parts of the boundary values of a function in H^p. That is

$$\psi(\eta) = \mathrm{Re}\big(f(\eta)\big) \quad ,$$

(3.45)

$$H\psi(\eta) = \mathrm{Im}\big(f(\eta)\big) \quad ,$$

for some $f \varepsilon H^p$.

Using this result, the solution of equations (3.38), (3.39) can be reduced to a Riemann-Hilbert problem in potential theory.

Rather than determining $\psi^*(x,t)$ directly we shall determine the partial derivatives, $\partial_x\psi^*(x,t)$ and $\partial_t\psi^*(x,t)$. We will later make assumptions that imply that ψ^* depends differentiably on x and t in the w-topology over the η-variable with derivatives in $L^p[0,1]$ for some p, $1 < p < 2$. This means that for every χ in $C[0,1]$ the function $(\chi,\psi^*(x,t))$ is a differentiable function of x and t, and there exist functions $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ in $L^p[0,1]$ such that

$$\partial_x(\chi,\psi^*) = (\chi,\psi_x^*) \; ,$$

(3.46)

$$\partial_t(\chi,\psi^*) = (\chi,\psi_t^*).$$

Whenever the support of χ lies inside the open sets $I^+(x,t)$ or $I^-(x,t)$ we see by (3.39) that the left hand side of (3.46) vanishes. Therefore

$$\psi_x^* = \psi_t^* = 0 \ \text{ off } \ I^0(x,t). \tag{3.47}$$

For any continuous function χ, $L\chi$ is also continuous and so, since L is a symmetric operator, we deduce that $(\chi, L\psi^*(x,t)) = (L\chi, \psi^*(x,t))$ is a differentiable function of x and t. By (3.46),

$$\begin{aligned} \partial_x(\chi, L\psi^*) &= (L\chi, \psi_x^*) = (\chi, L\psi_x^*), \\ \partial_t(\chi, L\psi^*) &= (L\chi, \psi_t^*) = (\chi, L\psi_t^*). \end{aligned} \tag{3.48}$$

Let I_0 be the interior of the closed set I^0 given by (3.36) in $[0,1]\times R\times R$ and let $I_0(x,t)$ denote its (x,t) slice defined as in (3.37). Choose the support of χ in (3.48) to lie inside the open set $I_0(x,t)$; then by (3.38), $L\psi^* = a$ on the support of χ:

$$(\chi, L\psi^*) = (\chi, a) .$$

Using the definition (1.23) of a we deduce that

$$\partial_x(\chi, L\psi^*) = (\chi, \eta), \quad \partial_t(\chi, L^*\psi) = -4(\chi, \eta^3). \tag{3.49}$$

Combining (3.48) and (3.49) we find

$$(3.50)\qquad L\psi_x^* = \eta \ , \quad L\psi_t^* = -4\eta^3 \quad \text{on} \quad I_0(x,t) \ .$$

By Theorem 3.15 both sides of (3.49) are continuous functions of η, so the equality may be extended to $\overline{I}_0(x,t)$, the closure of $I_0(x,t)$ in $[0,1]$.

Our later assumptions will imply that $I_0(x,t)$ and $I^0(x,t)$ differ by at most a finite number of points. Then $\overline{I}_0(x,t)$ and $I^0(x,t)$ will have a common interior, denoted $I(x,t)$, on which (3.50) holds. Since $I^0(x,t)$ and $I(x,t)$ differ by a set of measure zero, we can replace $I^0(x,t)$ by $I(x,t)$ in (3.47). We summarize:

ψ^* is in $L^p[0,1]$ for some p, $1 < p < \infty$, and satisfies

$$(3.51)\qquad \begin{aligned} L\psi_x^*(x,t) &= \eta \text{ on } I(x,t), \\ \psi_x^*(x,t) &= 0 \text{ off } I(x,t) \ . \end{aligned}$$

Similarly ψ_t^* is in $L^p[0,1]$ for some p, $1 < p < \infty$, and satisfies

$$(3.52)\qquad \begin{aligned} L\psi_t^* &= -4\eta^3 \quad \text{on} \quad I(x,t) \ , \\ \psi_t^* &= 0 \quad \text{off} \quad I(x,t) \ . \end{aligned}$$

Remark. Arguing as in Theorem 3.14, it follows that ψ_x^* and ψ_t^* are the unique L^p solution of (3.51) and (3.52) respectively.

We call (3.51) and (3.52) the differentiated variational conditions. Note that the inhomogeneous terms on the right (i.e. η and $-4\eta^3$) do not depend on the initial data $u(x)$.

If we extend $I(x,t)$ to the whole real line, R, by reflection about zero then (3.51) and (3.52) hold for the extended functions. We differentiate the top equations of (3.51) and (3.52) with respect to η on the open set $I(x,t)$; using (3.41) we obtain

$$H\psi_x^* = 1 \quad \text{on } I(x,t) , \tag{3.53}$$

and

$$H\psi_t^* = -12\eta^2 \quad \text{on } I(x,t) . \tag{3.54}$$

Denote by f the function in H^p whose real part is ψ_x^*:

$$\mathrm{Re}(f) = \psi_x^* . \tag{3.55}$$

We claim that

$$\begin{aligned} \mathrm{Im}(f) &= 1 \quad \text{on} \quad I(x,t) , \\ \mathrm{Re}(f) &= 0 \quad \text{off} \quad I(x,t) . \end{aligned} \tag{3.56}$$

The first relation is obtained by combining (3.53) and Theorem 3.16; the second relation is from (3.51).

Similarly, if g denotes the H^p function whose real part is ψ_t^*:

$$\text{(3.57)} \qquad \mathrm{Re}(g) = \psi_t^*$$

then

$$\mathrm{Im}(g) = -12\,\eta^2 \quad \text{on} \quad I(x,t)\ ,$$

$$\text{(3.58)}$$

$$\mathrm{Re}(g) = 0 \qquad \text{off } I(x,t)\ .$$

We shall exploit this fact in the next two sections to construct explicit solutions $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ of the differentiated variational conditions. We shall then verify that the function ψ^* satisfies the original variational condition.

We next show how to express $\bar{u}$, $\overline{u^2}$ and $\overline{u^3}$ in terms of ψ_x^* and ψ_t^*:

<u>Theorem 3.17.</u>

$$\bar{u}(x,t) = \frac{4}{\pi}\,(\eta,\psi_x^*(x,t)) \qquad ,$$

$$\text{(3.59)} \qquad \overline{u^2}(x,t) = \frac{4}{3\pi}\,(\eta,\psi_t^*(x,t)) = -\frac{16}{3\pi}\,(\eta^3,\psi_x^*(x,t))\ ,$$

$$\overline{u^3}(x,t) = -\frac{4}{3\pi}\,(\eta^3,\psi_t^*(x,t))\ .$$

Proof: We use formula (3.30) of Theorem 3.11:

$$\partial_x Q^*(x,t) = \frac{4}{\pi}\,(\eta,\psi^*(x,t)) \qquad ,$$

$$\partial_t Q^*(x,t) = -\frac{16}{\pi}\,(\eta^3,\psi^*(x,t))\ .$$

We use formula (3.46) to differentiate these with respect to x and t. We get

$$\partial_{xx}Q^*(x,t) = \frac{4}{\pi}\,(\eta,\psi_x^*(x,t)) \quad ,$$

$$\partial_{xt}Q^*(x,t) = \frac{4}{\pi}\,(\eta,\psi_t^*(x,t)) = -\frac{16}{\pi}\,(\eta^3,\psi_x^*(x,t)) \ , \tag{3.60}$$

$$\partial_{tt}Q^*(x,t) = -\frac{16}{\pi}\,(\eta^3,\psi_t^*(x,t)).$$

If we compare (3.60) with definitions (2.51), (2.54) and (2.56) for $\overline{u}$, $\overline{u^2}$ and $\overline{u^3}$ we conclude that (3.59) holds, proving the theorem.

We conclude this section by proving (2.29). Using the definition (1.25)of ϕ, the fact that L and $\frac{d}{d\eta}$ commute, and relation (3.41), we have

$$L\phi = -L\frac{d}{d\eta}\Phi = -\frac{d}{d\eta}L\Phi = H\Phi \quad .$$

According to Theorem 3.16, $L\phi$ is the imaginary part of the function of class H^p whose real part is Φ. But that function is

$$f(\eta) = \int_{-\infty}^{\infty} [(-u(y)-\eta^2)^{1/2} - i\eta]\, dy \ ,$$

as may be seen from (1.13).

Using (1.16) and (1.24) we see that

$$Im(f) = \theta_-(\eta) - \theta_+(\eta) \quad .$$

This proves (2.29).

4. The Solution until Breaktime.

The variational problem (2.16-2.18) contains x and t as parameters. We first investigate the case $t = 0$. We make the assumption, justified a posteriori, that the set I consists of a single interval

(4.1) $$I = (-\beta,\beta),$$

where β is a differentiable function of x.

We claim that the solution of the Riemann-Hilbert problem posed in (3.56) is

(4.2) $$f(\zeta) = i - \frac{\zeta}{(\beta^2-\zeta^2)^{1/2}},$$

where the sign of the square root is chosen so that

(4.3) $$i(\beta^2-\eta^2)^{1/2} > 0 \quad \text{for} \quad \eta > \beta.$$

Clearly $f(\zeta)$ is analytic in the upper half plane. On account of (4.3)

$$(\beta^2-\zeta^2)^{1/2} \sim -i\zeta \quad \text{for} \quad |\zeta| \quad \text{large},$$

which, by (4.2), shows that

(4.4) $$f(\zeta) = O\left(\frac{1}{|\zeta|^2}\right).$$

Since $f(\zeta)$ has singularities of the form $(\zeta-\beta)^{-1/2}$ on the real axis, we see from (4.4) that for $1 \leq p < 2$ there exists a costant $C > 0$ such that

$$\left(\int_{-\infty}^{\infty} |f(\eta+i\tau)|^p\right)^{1/p} < C$$

for all $\tau > 0$. This shows that f belongs to H^p for $1 \leq p < 2$.

For η real, it follows from (4.2) and (4.3) that

$$\text{(4.5)} \qquad \mathrm{Re}(f) = \begin{cases} -\dfrac{\eta}{(\beta^2-\eta^2)^{1/2}} & \text{on } I \\[2ex] 0 & \text{off } I, \end{cases}$$

and

$$\text{(4.6)} \qquad \mathrm{Im}(f) = \begin{cases} 1 & \text{on } I \\[2ex] 1 - \dfrac{\eta}{\eta^2-\beta^2)^{1/2}} & \text{off } I, \end{cases}$$

where the square root in (4.6) has the same sign as η. These relations show that conditions (3.56) are satisfied, Re(f) is odd with support in I, and Im(f) is even. We set (4.5) into (3.55):

$$\text{(4.7)} \qquad \psi_x^* = \begin{cases} -\dfrac{\eta}{(\beta^2-\eta^2)^{1/2}} & \text{for } |\eta| < \beta \\[2ex] 0 & \text{for } |\eta| \geq \beta. \end{cases}$$

Now using formula (3.41) of Theorem 3.15, with

$$H\psi_x^* = \mathrm{Im}(f)$$

given by (4.6), we have

$$(4.8)\qquad L\psi_x^* = \begin{cases} \eta & \text{for } 0 \leq \eta \leq \beta \\ \eta - \int_\beta^\eta \frac{\mu}{(\mu^2-\beta^2)^{1/2}}\, d\mu & \text{for } \eta > \beta \ . \end{cases}$$

Carrying out the integral in (4.8) we obtain

$$(4.9)\qquad L\psi_x^* = \begin{cases} \eta & \text{for } 0 \leq \eta \leq \beta \\ \eta - (\eta^2-\beta^2)^{1/2} & \text{for } \eta > \beta \ . \end{cases}$$

The value of β is so far undetermined; for that we turn to formula (3.59),

$$(4.10)\qquad \bar{u} = \frac{4}{\pi}\,(\eta,\ \psi_x^*).$$

We expect that

$$(4.11)\qquad \bar{u}(x,0) = u(x).$$

Combining (4.10) with (4.11) and expressing ψ_x^* in (4.10) by (4.7) yields

$$u(x) = \frac{4}{\pi}\,(\eta,\psi_x^*)$$

$$(4.12)\qquad = -\frac{4}{\pi}\int_0^\beta \frac{\eta^2}{(\beta^2-\eta^2)^{1/2}}\, d\eta \qquad = -\beta^2 \ ,$$

so we set $\beta(x) = \sqrt{-u(x)}$. Since $-u(x) > \eta^2$ between $x_-(\eta)$ and $x_+(\eta)$ and $-u(x) < \eta^2$ outside, see Figure 1, formula (4.7) may be rewritten as

$$\psi_x^*(x,0) = \begin{cases} -\dfrac{\eta}{(-u(x)-\eta^2)^{1/2}} & \text{for } x_-(\eta) < x < x_+(\eta) \\ 0 & \text{for } x \le x_-(\eta) \text{ or } x \ge x_+(\eta). \end{cases} \tag{4.13}$$

This completes the determination of $\psi_x^*(x,0)$.

To determine $\psi^*(x,0)$ itself we integrate $\psi_x^*(x,0)$ as given by (4.13). Theorem 3.13 states that $\psi^*(x,0) = 0$ where $\eta x - \theta_+(\eta) > 0$; in particular, it follows that for fixed η, $\psi^*(x,0) = 0$ for x large enough. Thus

$$\psi^*(x,0) = -\int_x^\infty \psi_x^*(y,0)\,dy \text{ , i.e.,}$$

$$\psi^*(x,0) = \begin{cases} 0 & \text{for } x \ge x_+(\eta) \\ \displaystyle\int_x^{x_+(\eta)} \frac{\eta}{(-u(y)-\eta^2)^{1/2}}\,dy & \text{for } x_-(\eta) < x < x_+(\eta) \\ \phi(\eta) & \text{for } x \le x_-(\eta). \end{cases} \tag{4.14}$$

In (4.14) we have used the definition (1.25) of $\phi(\eta)$:

$$\phi(\eta) = \int_{x_-(\eta)}^{x_+(\eta)} \frac{\eta}{(-u(y)-\eta^2)^{1/2}}\,dy \text{ .}$$

Clearly $0 \leq \psi^*(x,0) \leq \phi$ so $\psi^*(x,0)$ is in the admissible set A.

We shall now show that $\psi^*(x,0)$ given by (4.14) satisfies the variational conditions (3.34), thus justifying our assumption (4.1). Using (4.13) we may rewrite formula (4.9):

$$(4.15) \qquad L\psi_x^*(x,0) = \begin{cases} \eta & \text{for } x_-(\eta) \leq x \leq x_+(\eta) \\ \eta-(\eta^2+u(x))^{1/2} & \text{for } x < x_-(\eta) \text{ or } x > x_+(\eta). \end{cases}$$

It is easy to see from Theorem 3.13 that

$$\underset{x\to\infty}{\text{w-lim}} \ \psi^*(x,0) = 0 ,$$

so by Theorem 3.4 we see

$$\lim_{x\to\infty} L\psi^*(x,0) = 0 .$$

We then obtain by integrating (4.15)

$$- L\psi^*(x,0) = \int_x^\infty L\psi_x^*(y,0)\,dy$$

$$(4.16) \qquad = \begin{cases} \int_x^\infty \eta - (\eta^2+u(y))^{1/2}\,dy & \text{for } x > x_+(\eta) \\ -\eta x + \theta_+(\eta) & \text{for } x_-(\eta) \leq x \leq x_+(\eta) \\ -\eta x + \theta_+(\eta) - \int_x^{x_-(\eta)} (\eta^2+u(y))^{1/2}\,dy & \text{for } x \leq x_-(\eta) . \end{cases}$$

In (4.16) we have used the definition (1.16) of $\theta_+(\eta)$:

$$\theta_+(\eta) = \eta x_+(\eta) + \int_{x_+(\eta)}^\infty \eta - (\eta^2+u(y))^{1/2}\,dy.$$

Since $a(\eta,x,0) = \eta x - \theta_+(\eta)$, we see from (4.16) that

$$(4.17)\quad a-L\psi^{*}=\begin{cases}\displaystyle\int_{x_+(\eta)}^{x}(\eta^2+u(y))^{1/2}\,dy & \text{for } x > x_+(\eta)\\ 0 & \text{for } x_-(\eta)\le x\le x_+(\eta)\\ \displaystyle -\int_{x}^{x_-(\eta)}(\eta^2+u(y))^{1/2}\,dy & \text{for } x < x_-(\eta)\ ,\end{cases}$$

which clearly shows that $\psi^*(x,0)$ given by (4.14) satisfies the variational conditions (3.34).

According to Theorem 3.12, the only admissible function that satisfies the variational conditions is ψ^*. Thus we have proved

Theorem 4.1. $\psi^*(x,0)$ given by (4.14) solves the variational problem (2.16–2.18) at $t = 0$.

Formula (4.14)shows that $\psi^*(x,0)$ does indeed depend differentiably on x and $\psi^*_x(x,0)$ is given by (4.13). As shown in Theorem 3.17 the differentiability of ψ^* implies that (4.10) holds; comparing this with (4.13) we conclude from Theorem 2.12:

Theorem 4.2.

$$(4.18)\qquad \bar{u}(x,0) = \text{weak } L^2\text{-}\lim_{\varepsilon\to 0} u(x,0;\varepsilon) = u(x)\ .$$

This result was expected (see (4.11)) and provides justification for the replacement of the exact scattering data by the asymptotic data. In Theorem 4.5 we shall sharpen Theorem 4.2.

We now begin our investigation of nonzero values of t. If in the KdV equation one sets $\varepsilon = 0$,

$$(4.19)\qquad u_t - 6uu_x = 0$$

results. In the introduction we have pointed out that there is a breaktime, t_b, such that equation (4.19) has a smooth solution for $t < t_b$ but not beyond. Its value can be explicitly determined from the initial data $u(x)$:

$$t_b = [6 \max_x u_x(x)]^{-1}.$$

Denote by $u(x,t)$ the solution of (4.19) which takes on the prescribed initial values $u(x)$.

<u>Theorem 4.3</u>. For $0 \le t < t_b$

$$\bar{u}(x,t) = \text{weak } L^2\text{-}\lim_{\varepsilon\to 0} u(x,t;\varepsilon) = u(x,t) \tag{4.20}$$

uniformly in t.

Proof: We shall reduce this to Theorem 4.1. As $u(x,t)$ evolves according to (4.19) it will, until breaktime, remain of the same class as the initial data $u(x)$. We shall consider now how the asmptotic scattering data corresponding to $u(x,t)$ change with t. Define $x_+(\eta,t)$ and $x_-(\eta,t)$ as the two roots of

$$u(x_\pm,t) + \eta^2 = 0 \tag{4.21}$$

with $x_-(\eta,t) < x_+(\eta,t)$ for η in (0,1). By replacing $u(x)$ by $u(x,t)$ and $x_\pm(\eta)$ by $x_\pm(\eta,t)$ in formulas (1.12) and (1.16) we define

$$\Phi(\eta,t) = \int_{x_-(\eta,t)}^{x_+(\eta,t)} \left(-u(y,t)-\eta^2\right)^{1/2} dy \quad , \tag{4.22}$$

and

$$\theta_+(\eta,t) = \eta x_+(\eta,t) + \int_{x_+(\eta,t)}^{\infty} (\eta^2+u(y,t))^{1/2}\, dy. \tag{4.23}$$

<u>Lemma 4.4</u>. For $0 \le t < t_b$

$$x_\pm(\eta,t) = x_\pm(\eta) + 6\eta^2 t \quad , \tag{4.24}$$

(4.25) $$\Phi(\eta,t) = \Phi(\eta)$$

(4.26) $$\theta_+(\eta,t) = \eta_+(\eta) + 4\eta^3 t.$$

Proof: Differentiating (4.21) with respect to t and using (4.19) we obtain

$$\partial_t x_\pm = -\frac{u_t}{u_x} = -6u = 6\eta^2.$$

Formula (4.24) now follows upon integrating this and using the fact $x_\pm(\eta,0) = x_\pm(\eta)$. Differentiating (4.22) with respect to t and using (4.19) gives

$$\begin{aligned}\partial_t\Phi(\eta,t) &= -\frac{1}{2}\int_{x_-}^{x_+} \frac{u_t}{(-u-\eta^2)^{1/2}}\,dy \\ &= -3\int_{x_-}^{x_+} \frac{u}{(-u-\eta^2)^{1/2}}\,u_x\,dy \\ &= (2u - 4\eta^2)(-u-\eta^2)^{1/2}\Big|_{x_-}^{x_+} = 0\end{aligned}$$

by (4.21). This proves (4.25). Preforming the same procedure on (4.23) shows

$$\begin{aligned}\partial_t\theta_+(\eta,t) &= -\frac{1}{2}\int_{x_+}^{\infty} \frac{u_t}{(\eta^2+u)^{1/2}}\,dy \\ &= -3\int_{x_+}^{\infty} \frac{u}{(\eta^2+u)^{1/2}}\,u_x\,dy \\ &= (4\eta^2 - 2u)(\eta^2+u)^{1/2}\Big|_{x_+}^{\infty} = 4\eta^3\end{aligned}$$

which after integrating yields (4.26) and establishes Lemma 4.4.

From (4.25) of Lemma 4.4 we see that

$$\phi(\eta) = -\partial_\eta \Phi(\eta,t) \quad ,$$

and using (4.26) in the definition (1.23) of a, that

$$a(\eta,x,t) = \eta x - \theta_+(\eta,t).$$

Thus the variational problem (2.16-2.18) for $0 \leq t < t_b$ is the same as the variational problem for $t = 0$ except that $u(x)$ has been replaced by $u(x,t)$. By Theorem 4.1, its solution is given by formula (4.14) with the same replacements and Theorem 4.3 follows from Theorem 4.2.

The weak limit in formula (4.20) is in fact a strong limit.

<u>Theorem 4.5</u>. For $0 \leq t_0 < t_b$

$$\bar{u}(x,t) = L^2\text{-}\lim_{\varepsilon \to 0} u(x,t;\varepsilon) = u(x,t) \quad , \tag{4.27}$$

uniformly in t.

Remark. In particular this proves Theorem 1.2.

Proof: Lemma 2.13 gives

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} u^2(x,t,\varepsilon) = \int_{-\infty}^{\infty} u^2(x)\, dx \quad ,$$

uniformly in t. Since $\int u^2\, dx$ is a conserved quantity for equation (4.19),

$$(4.28) \qquad \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} u^2(x,t;\varepsilon)\, dx = \int_{-\infty}^{\infty} u^2(x,t)\, dx \quad ,$$

uniformly in t for $0 \leq t < t_b$.

By Theorem 4.3

$$(4.29) \qquad \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} w(x)u(x,t;\varepsilon)\, dx = \int_{-\infty}^{\infty} w(x)u(x,t)\, dx \ ,$$

uniformly in t for $0 \leq t < t_b$ and over w in any compact subset of $L^2(R)$. But it is easily seen that for $0 \leq t \leq t_b$, $u(x,t)$ is a continuous function of t into $L^2(R)$ of the x variable. Thus the set $\{u(x,t):\ 0 \leq t \leq t_b\}$ is a compact subset of $L^2(R)$; setting $w(x) = u(x,t)$ in (4.29) we conclude that

$$(4.30) \qquad \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} u(x,t)u(x,t;\varepsilon)\, dx = \int_{\infty}^{\infty} u^2(x,t)\, dx \quad ,$$

uniformly in t for $0 \leq t < t_b$.

Consider the identity

$$\int_{-\infty}^{\infty} \left(u(x,t;\varepsilon) - u(x,t)\right)^2 dx = \int_{-\infty}^{\infty} u^2(x,t;\varepsilon)\, dx - 2\int_{-\infty}^{\infty} u(x,t)\, u(x,t;\varepsilon)\, dx + \int_{-\infty}^{\infty} u^2(x,t)\, dx \ .$$

By (4.28) and (4.30) the righthand side tends to zero uniformly over $0 \leq t < t_b$ as ε tends to zero, proving the theorem.

In the proof of Theorem 4.3 we relied on the knowledge that $\bar{u}(x,t) = u(x,t)$ satisfies (4.19) for $0 \leq t < t_b$. We show now how (4.19) can be deduced directly from the variational conditions.

Assume that the set I consists of a single interval, see (4.1), where β is a differentiable function of x and t. This assumption will be justified a posteriori. We claim that the solution of the Riemann-Hilbert problem posed in (3.58) of Theorem 3.18 is

$$g(\zeta) = -12\,\zeta^2 i + \frac{12\zeta^3 - 6\beta^2\zeta}{(\beta^2-\zeta^2)^{1/2}} , \tag{4.31}$$

where the sign of the square root is again chosen to satisfy (4.3). The same arguments that applied to $f(\zeta)$ given by (4.2) can now be applied to $g(\zeta)$ to show g belongs to H^p for $1 \leq p < 2$, and for η real

$$\operatorname{Re}(g) = \begin{cases} \dfrac{12\eta^3 - 6\beta^2\eta}{(\beta^2-\eta^2)^{1/2}} & \text{on } I \\ \\ 0 & \text{off } I , \end{cases} \tag{4.32}$$

and

$$\operatorname{Im}(g) = \begin{cases} -12\,\eta^2 & \text{on } I \\ \\ -12\,\eta^2 + \dfrac{12\eta^3 - 6\beta^2\eta}{(\eta^2-\beta^2)^{1/2}} & \text{off } I . \end{cases} \tag{4.33}$$

As before, the square root in (4.33) has the same sign as η. These relations show that conditions (3.58) are satisfied, Re(g) is odd, with support in I, and Im(g) is even.

We set (4.32) into (3.57) to obtain

$$\psi_t^* = \begin{cases} \dfrac{12\eta^3-6\beta^2\eta}{(\beta^2-\eta^2)^{1/2}} & \text{for } |\eta| < \beta \\ 0 & \text{for } |\eta| \geq \beta. \end{cases} \tag{4.34}$$

Now using formula (3.41) of Theorem 3.15 with

$$H\psi_t^* = \operatorname{Im}(g)$$

given by (4.33) we have

$$L\psi_t^* = \begin{cases} -4\eta^3 & \text{for } 0 \leq \eta \leq \beta \\ -4\eta^3 + \displaystyle\int_\beta^\eta \frac{12\mu^3-6\beta^2\mu}{(\mu^2-\beta^2)^{1/2}}\,d\mu & \text{for } \eta > \beta. \end{cases} \tag{4.35}$$

Carrying out the integral in (4.35) we obtain

$$L\psi_t^* = \begin{cases} -4\eta^3 & \text{for} \quad 0 \leq \eta \leq \beta \\ -4\eta^3 + (4\eta^2+2\beta^2)(\eta^2-\beta^2)^{1/2} & \text{for} \quad \eta > \beta \;. \end{cases} \tag{4.36}$$

Now ψ_x^* given by (4.7) and ψ_t^* given by (4.34) must satisfy the compatibility condition $\partial_t\psi_x^* = \partial_x\psi_t^*$. This implies

$$\begin{aligned} 0 &= \partial_t\left(\frac{\eta}{(\beta^2-\eta^2)^{1/2}}\right) + \partial_x\left(\frac{12\eta^3-6\beta^2\eta}{(\beta^2-\eta^2)^{1/2}}\right) \\ &= -\frac{\eta\beta}{(\beta^2-\eta^2)^{3/2}}\left(\beta_t + 6\beta^2\beta_x\right) \;, \end{aligned} \tag{4.37}$$

from which we conclude

$$\beta_t + 6\,\beta^2\beta_x = 0 \;. \tag{4.38}$$

Expressing ψ_x^* in formula (4.10) by (4.7), we obtain (see 4.12)

$$\bar{u}(x,t) = \frac{4}{\pi}\,(\eta,\psi_x^*) = -\,\beta^2(x,t) \;. \tag{4.39}$$

Multiplying (4.38) by -2β and using (4.39) we deduce

$$\bar{u}_t - 6\bar{u}\bar{u}_x = 0 \;; \tag{4.40}$$

this is relation (4.19), valid for $t \leq t_b$.

The above procedure will be generalized in the next section to

obtain the time dependence of solutions of the variational problem for $t > t_b$.

We close this section with an observation. Set ψ_x^* given by (4.7) and ψ_t^* given by (4.34) into (3.59) of Theorem 3.17:

$$\overline{u^2}(x,t) = -\frac{16}{3\pi}(\eta^3,\psi_x^*) = \frac{4}{3\pi}\int_0^\beta \frac{4}{(\beta^2-\eta^2)^{1/2}}\,d\eta ,$$

$$\overline{u^3}(x,t) = -\frac{4}{3\pi}(\eta^3,\psi_t^*) = -\frac{4}{3\pi}\int_0^\beta \frac{12\eta^6-6\beta^2\eta^4}{(\beta^2-\eta^2)^{1/2}}\,d\eta .$$

A brief calculation shows that

$$\overline{u^2}(x,t) = \beta^4(x,t) = \bar{u}^2(x,t) ,$$

(4.41)

$$\overline{u^3}(x,t) = -\beta^6(x,t) = \bar{u}^3(x,t) .$$

From the definitions (2.51), (2.54) and (2.56) for $\bar{u}$, $\overline{u^2}$ and $\overline{u^3}$ we must have

$$\bar{u}_t - 3\,\overline{u^2}_x = 0 ,$$

(4.42)

$$\overline{u^2}_t - 4\,\overline{u^3}_x = 0 .$$

Using (4.41) to express $\overline{u^2}$ and $\overline{u^3}$ in terms of $\bar{u}$, we see each of (4.42) will hold if and only if (4.40) holds, showing again the validity of (4.40).

5. <u>The Solution beyond Breaktime.</u>

In this section we describe the solution of the variational problem (2.16-2.18) when t is larger than the breaktime, t_b . We make the assumption that the set I consists of a finite union of disjoint intervals, say

$$(5.1) \qquad I = (\beta_{2n},\beta_{2n-1}) \quad \cdots \quad (\beta_2,\beta_1) \ ,$$

where

$$(5.2) \qquad \beta_{2n} < \beta_{2n-1} < \cdots < \beta_2 < \beta_1 \ .$$

Since I is symmetric about $\eta = 0$, we see that

$$\beta_{2n-k+1} = - \beta_k \ ,$$

and that the set I is completely determined by its set of positive end points

$$(5.3) \qquad B = \{\beta_k\}_{k=1}^{n} \ .$$

We will work as if n is odd since, as we will later see, this will be the case except when the initial function u(x) takes on the value zero; in that case we make n odd by setting $\beta_n = 0$.

As first step we shall construct all possible solutions of the Riemann-Hilbert problems (3.56) and (3.58) that ψ_x^* and ψ_t^* satisfy. The solution will be expressed in terms of a function $R(\zeta)$ defined by

$$(5.4) \qquad R^2(\zeta) = \prod_{k=1}^{n} (\beta_k{}^2 - \zeta^2) \ ;$$

we choose its sign so that

(5.5) $$iR(\eta) > 0 \quad \text{for} \quad \eta > \beta \ .$$

Lemma 5.1. All functions of the form

(5.6) $$f(\zeta) = i - \frac{P(\zeta)}{R(\zeta)} \quad ,$$

(5.7) $$g(\zeta) = -12\zeta^2 i + \frac{Q(\zeta)}{R(\zeta)} \ ,$$

are solutions of the Riemann-Hilbert problems (3.56) and (3.58) respectively, where $P(\zeta)$ and $Q(\zeta)$ are odd polynomials with real coefficients

(5.8) $$P(\zeta) = \zeta^n + a_2\zeta^{n-2} + \dots + \alpha_{n-2}\zeta$$

(5.9) $$Q(\zeta) = 12\zeta^{n+2} - 6(\Sigma\,\beta_k^2)\zeta^n + \gamma_2\zeta^{n-2} + \dots + \gamma_{n-2}\zeta \ .$$

Remark. It is not hard to show that these are all solutions [9].

Proof: Clearly $f(\zeta)$ and $g(\zeta)$ are analytic in the upper half plane. By (5.4), (5.5)

(5.10) $$iR(\zeta) \sim \zeta^n - \frac{1}{2}\Big(\sum_{k=1}^{n} \beta_k^2\Big)\zeta^{n-2} + 0(|\zeta|^{n-4})$$

for $|\zeta|$ large, which, along with (5.6-5.9), shows that

(5.11) $$f(\zeta) = 0\Big(\frac{1}{|\zeta|^2}\Big), \qquad g(\zeta) = 0\Big(\frac{1}{|\zeta|^2}\Big) \ .$$

Since $f(\zeta)$ and $g(\zeta)$ have singularities of the form $(\zeta-\beta)^{1/2}$ on the real axis, we see from (5.11) that f and g belong to H^p for $1 \leq p < 2$.

For η real, the function $R(\eta)$ defined by (5.4) is real and even on I, imaginary and odd off I, where I is defined by (5.1). Since $P(\eta)$ and $Q(\eta)$ are real and odd for η real, it follows from (5.6) and (5.7) that:

$$\text{(5.12)} \qquad \operatorname{Re}(f) = \begin{cases} -\dfrac{P(\eta)}{R(\eta)} & \text{on } I \\ 0 & \text{off } I, \end{cases}$$

and

$$\text{(5.13)} \qquad \operatorname{Im}(f) = \begin{cases} 1 & \text{on } I \\ 1 + i\,\dfrac{P(\eta)}{R(\eta)} & \text{off } I, \end{cases}$$

while

$$\text{(5.14)} \qquad \operatorname{Re}(g) = \begin{cases} \dfrac{Q(\eta)}{R(\eta)} & \text{on } I \\ 0 & \text{off } I, \end{cases}$$

and

$$(5.15) \qquad \mathrm{Im}(g) = \begin{cases} -12\,\eta^2 & \text{on } I \\ -12\,\eta^2 - i\,\dfrac{Q(\eta)}{R(\eta)} & \text{off } I\,. \end{cases}$$

These relations show that $f(\zeta)$ and $g(\zeta)$ satisfy conditions (3.56) and (3.58) as asserted in Lemma 5.1.

Using (3.55) and (3.57) we set

$$(5.16) \qquad \psi_x^* = -\mathrm{Re}\left(\frac{P(\eta)}{R(\eta)}\right) ,$$

$$(5.17) \qquad \psi_t^* = \mathrm{Re}\left(\frac{Q(\eta)}{R(\eta)}\right) .$$

We now show how to choose the coefficients α_j and γ_j so that the differentiated variational conditions, (3.51) and (3.52), are satisfied. Using formula (3.41) with

$$H\psi_x^* = \mathrm{Im}(f)$$

given by (5.13) and with

$$H\psi_t^* = \mathrm{Im}(g)$$

given by (5.15), we obtain

$$(5.18) \qquad L\psi_x^* = \eta - \int_0^\eta \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\mu \quad ,$$

and

$$(5.19) \qquad L\psi_t = -4\eta^3 + \int_0^\eta \mathrm{Im}\left(\frac{Q(\mu)}{R(\mu)}\right) d\mu \;.$$

If the differentiated variational conditions, (3.51) and (3.52) are to be satisfied, the integrals on the righthand sides of (5.18) and (5.19) must vanish on I. Since the integrands are even functions which vanish on I, we see it is both necessary and sufficient to require

$$(5.20)\qquad \int_{\beta_{2k+1}}^{\beta_{2k}} \frac{P(\eta)}{R(\eta)}\, d\eta = 0 \quad ,$$

$$(5.21)\qquad \int_{\beta_{2k+1}}^{\beta_{2k}} \frac{Q(\eta)}{R(\eta)}\, d\eta = 0 \quad ,$$

for $k = 1,2,\ldots,\frac{n-1}{2}$. Recalling the form of P and Q given in (5.8) and (5.9), we see that (5.20) is a system of $(n-1)/2$ inhomogeneous linear equations for the $(n-1)/2$ unknowns $\alpha_2,\alpha_4,\ldots,\alpha_{n-1}$ and (5.21) is a system of $(n-1)/2$ inhomogeneous linear equations for the $(n-1)/2$ unknowns $\gamma_2,\gamma_4,\ldots,\gamma_{n-1}$.

Lemma 5.2. The systems of equations (5.20) and (5.21) have unique solutions.

First Proof: If not, then according to the alternative principle of linear algebra, the common corresponding homogeneous system would have a nontrivial solution. But then there would be a nontrivial odd polynomial $N(\eta)$ with real coefficients and of degree at most $n-2$ that satisfies

$$(5.22)\qquad \int_{\beta_{2k+1}}^{\beta_{2k}} \frac{N(\eta)}{R(\eta)}\, d\eta = 0$$

for $k = 1,2,\ldots,\frac{n-1}{2}$. This implies

$$\psi_0 = \text{Re}\left(\frac{N(\eta)}{R(\eta)}\right)$$

satisfies

$$L\psi_0 = 0 \qquad \text{on } I$$

$$\psi_0 = 0 \qquad \text{off } I\ .$$

According to the uniqueness argument given for (3.40), such a ψ_0 is zero, and so $N(\eta)$ is zero, a contradiction.

Second Proof: If not, then let $N(\eta)$ satisfy (5.22) as above. But this contradicts the following lemma.

Lemma 5.3. Let $N(\eta)$ be any nontrivial odd polynomial with real coefficients that satisfies (5.22). Then $N(\eta)$ must have at least one root in each of the $(n-1)/2$ intervals $(\beta_{2k+1},\beta_{2k})$ and the degree of $N(\eta)$ must be at least n.

Proof: Since $R(\eta)$ does not change sign in $(\beta_{2k+1},\beta_{2k})$, then $N(\eta)$ must if (5.22) is to hold. Thus $N(\eta)$ has at least one positive root in each of the $(n-1)/2$ intervals $(\beta_{2k+1},\beta_{2k})$. Since N is odd it has an equal number of negative roots and a root at zero, altogether a minimum of n roots, proving the lemma.

According to (5.20), the polynomial $P(\eta)$ satisfies hypothesis (5.22) of Lemma 5.3. Since P is of degree n, we conclude

Corollary 5.4. $P(\eta)$ has a zero inside each of the intervals $(\beta_{2k+1},\beta_{2k})$, $k = 1,\ldots,n-1$ and at $\eta = 0$, and nowhere else.

According to Lemma 5.2, we can solve (5.20) and (5.21) uniquely for α_j and γ_j , $j = 2,4,\ldots,n-1$ as functions of the points β_k in B, see (5.3). Observe that the coefficients and the inhomogeneous terms appearing in the linear systems (5.20) and (5.21) are complete

hyperelliptic integrals. Thus the α_j and γ_j are ratios of determinants whose entries are such integrals.

We have now proved

Theorem 5.5. Given the set B, ψ_x^* as determined by (5.16) and (5.20) is the solution of the differential variational conditions (3.51) and ψ_t^* as determined by (5.17) and (5.21) is the solution of the differentiated variational condition (3.52).

The functions ψ_x^* and ψ_t^* satisfy the compatibility condition $\partial_t\psi_x^* = \partial_x\psi_t^*$; or equivalently

$$\partial_t \frac{P(\zeta)}{R(\zeta)} + \partial_x \frac{Q(\zeta)}{R(\zeta)} = 0 \,. \tag{5.23}$$

We shall see what conditions this imposes on the set B.

Rewrite the left side (5.23) as

$$\partial_t \frac{P(\zeta)}{R(\zeta)} + \partial_x \frac{Q(\zeta)}{R(\zeta)} = \frac{R^2(\partial_t P + \partial_x Q) - \frac{1}{2}(P\partial_t R^2 + Q\,\partial_x R^2)}{R^3} \,. \tag{5.24}$$

We claim that the numerator on the right is an odd polynomial of degree at most 3n-2. For from the definitions (5.4), (5.8) and (5.9) we see that for large $|\zeta|$

$$\begin{aligned} R^2(\zeta) &= (-\zeta^2)^n + \Big(\sum_{k=1}^{n} \beta_k^2\Big)(-\zeta^2)^{n-1} + O(|\zeta|^{2n-4}) \,, \\ P(\zeta) &= \zeta^n + O(|\zeta|^{n-2}) \,, \\ Q(\zeta) &= 12\,\zeta^{n+2} - 6\Big(\sum_{k=1}^{n} \beta_k^2\Big)\zeta^n + O(|\zeta|^{n-2}) \,, \end{aligned}$$

so that

(5.25) $$R^2(\partial_t P + \partial_x Q) = (-\zeta^2)^n\, \partial_x\Big(-6 \sum_{k=1}^{n} \beta_k^2\Big)\zeta^n + O(|\zeta|^{3n-2}) \;,$$

and

(5.26) $$\frac{1}{2}\,(P\partial_t R^2 + Q\partial_x R^2) = \frac{1}{2}\,(12\zeta^{n+2})\; \partial_x\Big(\sum_{k=1}^{n} \beta_k^2\Big)(-\zeta^2)^{n-1} + O(|\zeta|^{3n-2}).$$

Subtracting (5.26) from (5.25) we find the leading terms on the right cancel, demonstrating the claim.

Since $R^2 = 0$ at $\zeta = \beta_k$, the numerator on the right of (5.24) vanishes at $\zeta = \beta_k$ if and only if its second term vanishes at $\zeta = \beta_k$. That second term may be written

$$\frac{1}{2}\,(P\partial_t R^2 + Q\partial_x R^2) = \sum_{k=1}^{n} \Big[\beta_k\big(P(\zeta)\partial_t\beta_k + Q(\zeta)\partial_x\beta_k\big) \prod_{j\neq k} (\beta_j^2 - \zeta^2)\Big] \;.$$

Clearly this vanishes at $\zeta = \beta_k$ if and only if β_k satisfies

(5.27) $$P(\beta_k)\, \partial_t\beta_k + Q(\beta_k)\partial_x\beta_k = 0 \;.$$

This time evolution of the β_k allows us to factor $R^2(\zeta)$ from the numerator of (5.24) and write

(5.28) $$\partial_t \frac{P(\zeta)}{R(\zeta)} + \partial_x \frac{Q(\zeta)}{R(\zeta)} = \frac{N(\zeta)}{R(\zeta)} \;,$$

where $N(\zeta)$ is the odd polynomial of degree at most n-2 given by

$$N(\zeta) = \partial_t P(\zeta) + \partial_x Q(\zeta) + \sum_{k=1}^{n} \Big[\frac{P(\zeta)-P(\beta_k)}{\zeta^2-\beta_k^2}\,\beta_k\partial_t\beta_k + \frac{Q(\zeta)-Q(\beta_k)}{\zeta^2-\beta_k^2}\,\beta_k\partial_x\beta_k \Big] \;.$$

In fact, we will show $N(\zeta) = 0$.

For ε sufficiently small we employ the fundamental theorem of calculus to obtain

$$\partial_t \int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{P(\eta)}{R(\eta)}\, d\eta + \partial_x \int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{Q(\eta)}{R(\eta)}\, d\eta$$

$$(5.29) \quad = \int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \partial_t \frac{P(\eta)}{R(\eta)} + \partial_x \frac{Q(\eta)}{R(\eta)}\, d\eta$$

$$+ \frac{P(\beta_{2k}-\varepsilon)\, \partial_t\beta_{2k} + Q(\beta_{2k}-\varepsilon)\, \partial_x\beta_{2k}}{R(\beta_{2k}-\varepsilon)} ,$$

$$- \frac{P(\beta_{2k+1}+\varepsilon)\, \partial_t\beta_{2k+1} + Q(\beta_{2k+1}+\varepsilon)\, \partial_x\beta_{2k+1}}{R(\beta_{2k+1}+\varepsilon)}$$

for $k = 1,2,\ldots,(n-1)/2$. As ε goes to zero, the denominators of the second and third terms on the right of (5.29) will, by (5.4), vanish like $\varepsilon^{1/2}$, but, by (5.27), the numerators of the same terms are $O(\varepsilon)$; thus the limit of those terms as $\varepsilon \to 0$ is zero. Using (5.28) in the first term, we then obtain

$$(5.30) \quad \lim_{\varepsilon\to 0} \left[\partial_t \int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{P(\eta)}{R(\eta)} d\eta + \partial_x \int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{Q(\eta)}{R(\eta)}\, d\eta\right] = \int_{\beta_{2k+1}}^{\beta_{2k}} \frac{N(\eta)}{R(\eta)}\, d\eta \quad ,$$

uniformly on compact subsets of x and t. On the other and, by (5.20) and (5.21),

$$(5.31)\qquad \lim_{\varepsilon\to 0}\int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{P(\eta)}{R(\eta)}\,d\eta = 0\ ,\qquad \lim_{\varepsilon\to 0}\int_{\beta_{2k+1}+\varepsilon}^{\beta_{2k}-\varepsilon} \frac{Q(\eta)}{R(\eta)}\,d\eta = 0\ ,$$

uniformly on compact subsets of x and t. Since the divergence is a closed operator, we conclude from (5.30) and (5.31) that

$$\int_{\beta_{2k+1}}^{\beta_{2k}} \frac{N(\eta)}{R(\eta)}\,d\eta = 0\ .$$

Using Lemma 5.3, we then conclude $N(\eta) = 0$. Thus we have shown

<u>Theorem 5.6</u>. If the β_k in B satisfy (5.27), then ψ_x^* and ψ_t^* given by (5.16) and (5.17) are compatible.

We deduce from Corollary 5.4 that

$$P(\beta_k) \neq 0\ , \qquad k = 1,\ldots,n\ .$$

Dividing (5.27) by $P(\beta_k)$ gives

$$(5.32)\qquad \partial_t\beta + V(\beta;B)\,\partial_x\beta \quad \text{for } \beta \text{ in } B,$$

where

$$V(\eta;B) = \frac{Q(\eta)}{P(\eta)}$$

indicates the dependence of V on the set B through the dependence of α_j and γ_j on B.

Formula (5.32) is a coupled system of n partial differential

equations where each β in B is a Riemann invariant with corresponding characteristic velocity $V(\beta;B)$. The set B evolves according to (5.32) so long as the solutions remain regular and n does not change. We have not yet said how the set B, and its cardinality, change when a sigularity is encountered, nor have we addressed how $B(x,t)$ is related to the initial data $u(x)$. We shall deal with these issues by introducing a pair of functions $x_+(\eta,t)$ and $x_-(\eta,t)$, $0 \leq \eta \leq 1$, already defined in Section 4 for $t < t_b$ but here extended to all t ; we shall construct the set B in terms of these functions.

We start by defining the notion of crossing:

Definition. A pair of functions $x_+(\cdot)$, $x_-(\cdot)$ cross the value x at η if the union of the images of any neighborhood of η under $x_+(\cdot)$ and $x_-(\cdot)$ is a neighborhood of x. Such an η is called a crossing point of the pair at x.

Note that if $0 < \eta < 1$ and $x_\pm(\eta) = x$, then η is a crossing point at x unless $x_\pm(\cdot)$ has a local extremum at η.

For a given pair of functions $x_+(\eta)$, $x_-(\eta)$ and a given value x we define the set B(x) by

$$B(x) = \{\eta : \eta \text{ is a crossing point of } x_+(\cdot),\ x_-(\cdot) \text{ at } x\} .$$

Note that if $u(x)$ is our initial function and $x_\pm(\eta)$ are defined by (1.11), then B(x) consists of the single point $\beta(x)$ defined by (4.13):

$$\beta(x) = \left(-u(x)\right)^{1/2}.$$

Suppose the functions $x_{\pm} = x_{\pm}(\cdot,t)$ depend on the parameter t; then so does the set B defined above:

(5.33) $$B(x,t) = \{\eta;\ \eta \text{ is a crossing point of } x_+(\cdot,t),\ x_-(\cdot,t) \text{ at } x\}$$

We formulate the following initial value problem for the functions $x_+(\eta,t)$, $x_-(\eta,t)$:

(5.34) $$\begin{aligned} \frac{dx_+}{dt} &= V(\eta;B(x_+,t)) , \quad x_+(\eta,0) = x_+(\eta) \\ \frac{dx_-}{dt} &= V(\eta;B(x_-,t)) , \quad x_-(\eta,0) = x_-(\eta), \end{aligned}$$

where $x_+(\eta)$ and $x_-(\eta)$ are defined by (1.11). Note that the set $B(x,t)$ depends on the functions x_+ ,x_-. As long as that dependence is Lipschitz continuous in a suitable norm for x_+, x_-, (5.34) has a unique solution. We assume that (5.34) has unique solutions such that

(a) As functions of η and t, x_+ and x_- re C^1 in $(0,1) \times R$ and continuous in $(0,1] \times R$.

(b) The limits $\lim_{\eta\to 0} x_{\pm}(\eta,t)$ exist, possibly as $\pm\infty$. If finite we call it the boundary value of $x_{\pm}$ at $\eta = 0$, denote it by $x_{\pm}(0,t)$ and assume it is C^1 in t.

(c) The number of critical points of $x_+(\cdot,t)$ and $x_-(\cdot,t)$ is finite for all t.

(d) If for some η, $x_+(\eta) = x_-(\eta)$, then $x_+(\eta,t) = x_-(\eta,t)$ for all t.

Note that (c) implies that the cardinality of $B(x,t)$ is finite. Since $x_+(1) = x_-(1)$, it follows that $x_+(1,t) = x_-(1,t)$; we denote this

common value by $x_0(t)$. From (d) and the fact that $x_-(\eta) < x_+(\eta)$ for $0 < \eta < 1$ we conclude that $x_-(\eta,t) < x_+(\eta,t)$ for all t as well.

Note that according to the definition of crossing, $\eta = 1$ is a crossing point at $x = x_0(t)$ iff for η near 1

$$x_-(\eta,t) < x_0(t) < x_+(\eta,t).$$

<u>Lemma 5.7</u>. Suppose that $\bar{x}$ is not a critical value of $x_+(\cdot,\bar{t})$ or $x_-(\cdot,\bar{t})$, nor a boundary value of $x_\pm(\cdot,\bar{t})$ (i.e., $\bar{x} \neq x_\pm(0,\bar{t})$ or $\bar{x} \neq x_0(t)$) then for all x,t in a neighborhood of $\bar{x},\bar{t}$

(a) The cardinality of B(x,t) is constant

(b) B(x,t) is a differentiable function of x,t.

Proof: It follows from our assumptions that we can find a neighborhood of $\bar{x},\bar{t}$ such that $\partial_\eta x_\pm$ are bounded away from zero and x is bounded away from the boundary values of $x_\pm(\cdot,t)$. By the implicit function theorem, there exists a (maybe smaller) neighborhood N of $\bar{x},\bar{t}$ such that every solution of $x = x_\pm(\eta,t)$ is given by $\eta = \beta(x,t)$ where $\beta(\cdot,\cdot)$ is a C^1 function in N. The conclusions of Lemma 5.7 readily follows.

Now construct I(x,t) from B(x,t) according to (5.1).

<u>Lemma 5.8</u>. For any x,t and any η in (0,1)

$$x_-(\eta,t) < x < x_+(\eta,t) \Rightarrow \eta \in I(x,t) , \tag{5.35}$$

$$x < x_-(\eta,t) \quad \text{or} \quad x > x_+(\eta,t) \Rightarrow \eta \notin I(x,t) . \tag{5.36}$$

Proof: By (5.1) if η is not in B(x,t) then η is in I(x,t) if and

only if the number of elements of $B(x,t)$ in $[\eta,1]$ is odd. If $x \neq x_0(t)$ then by (5.33) the number of elements of $B(x,t)$ in $[\eta,1]$ is equal to the number of times that $x_+(\mu,t)$ or $x_-(\mu,t)$ crosses the value x for μ in $[\eta,1]$. Consider the case $x_-(\eta,t) < x < x_+(\eta,t)$; if $x < x_0(t) = x_\pm(1,t)$ then $x_+(\mu,t)$ must cross the value x at an even number of μ in $[\eta,1]$ while $x_-(\mu,t)$ must cross the value x at an odd number of μ in $[\eta,1]$. Thus, the total number of crossings beings odd we conclude $\eta \in I(x,t)$. The argument is essentially the same for all the cases, provided $x \neq x_0(t)$. In that case one must consider the behavior of x_+ and x_- near $\mu = 1$ but it is just as easy.

Now we use formulas (5.16) and (5.17) to construct $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ from $B(x,t)$.

Lemma 5.9. If $\bar{x}$ is not a critical or boundary value of $x_+(\cdot,\bar{t})$ or $x_-(\cdot,\bar{t})$ then in a neighborhood of $\bar{x},\bar{t}$

(a) $\psi_x^*(x,t)$ and $\psi_t^*(x,t)$ are differentiable functions of x,t.

(b) The compatibility condition

$$\partial_t\psi_x^* = \partial_x\psi_t^*$$

is satisfied.

Proof: The coefficients α_j and γ_j of P and Q are functions of $\beta_1,\ldots,\beta_n$, expressible as rational functions of complete hyperelliptic integrals. Therefore they are analytic functions of $\beta_1,\ldots,\beta_n$ except if two β_k are equal. It follows from Lemma 5.7 that this does not happen under the hypothesis of Lemma 5.9. It does not follows from Lemma 5.7 that the β_k are differentiable functions of x,t. Part (a) follows from these two observations.

To prove part (b), we differentiate with respect to x, and then t, the relation

$$x = x_{\pm}(\beta, t) \ , \quad \beta = \beta(x,t) \ ;$$

we get

$$1 = \partial_\eta x_\pm \, \partial_x \beta \ , \quad 0 = \partial_\eta x_\pm \, \partial_t \beta + \partial_t x_\pm \ .$$

Using equation (5.34) to express $\partial_t x_\pm$ as V we obtain,

$$\partial_t \beta = - \partial_t x_\pm / \partial_\eta x_\pm = - V \partial_x \beta \quad .$$

This is equation (5.32), equivalent to (5.27). We appeal now to Theorem 5.6 to conclude that (b) holds.

<u>Lemma 5.10</u>. For $\eta > 0$

$$\text{(5.37)} \qquad \mathrm{Re}\left(\frac{P(\eta)}{R(\eta)} \right) \geq 0 \quad ,$$

$$\text{(5.38)} \qquad \int_0^\eta \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)} \right) d\mu \geq 0 \ .$$

Proof: By (5.12), $\mathrm{Re}\left(\frac{P(\eta)}{R(\eta)}\right)$ is nonzero only on the set I. By Corollary 5.4, $P(\eta)$ alternates sign on the intervals that make up I, and by (5.8) it is positive in the rightmost interval. On the other hand, by (5.4) and (5.5), $R(\eta)$ has the same property, so (5.37) follows.

Condition (5.20) ensures that $\int_0^\eta \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\mu$ is nonzero only off the set I. By Corollary 5.4, we see that $P(\mu)$ alternatively crosses from negative to positive and positive to negative on the intervals $(\beta_{2k+1}, \beta_{2k})$ in such a way that it is positive on (β_1, ∞). Since, by (5.4) and (5.5), $iR(\mu)$ alternates sign on those intervals and is

positive on (β_1,∞), we conclude that $\operatorname{Im}\left(\frac{P(\mu)}{R(\mu)}\right)$ crosses from positive to negative on the intervals $(\beta_{2k+1},\beta_{2k})$ and is positive on (β_1,∞). From this fact (5.38) follows, completing the proof of the lemma.

To construct the solution of the variational problem (2.16-2.18), $\psi^*(x,t)$, we integrate $\psi^*_x(x,t)$ as given by (5.16). Theorem 3.13 requires that $\psi^*(x,t) = 0$ where $\eta x - 4\eta^3 t - \theta_+(\eta) > 0$; in particular, it follows that, for fixed η and t, $\psi^*(x,t) = 0$ for x large enough. Thus we set

$$(5.39)\quad \psi^*(x,t) = -\int_x^\infty \psi^*_x(y,t)\,dy \equiv \int_x^\infty \operatorname{Re}\left(\frac{P(\eta)}{Q(\eta)}\right)dy\,.$$

Lemma 5.8 shows that, for fixed η and t, the integrand is zero for x large. Lemma 5.9 shows that $\partial_t\psi^*(x,t)$ as calculated from (5.39) equals $\psi^*_t(x,t)$ as given by (5.17). Inequality (5.37) of Lemma 5.10 shows that ψ^* is a positive and decreasing function of x. In the x,t region where $x < x_-(\eta,t)$, Lemma 5.8 shows that

$$\psi^* = \int_{x_-(\eta,t)}^{x_+(\eta,t)} \frac{P(\eta)}{R(\eta)}\,dy \quad ,$$

and

$$\partial_x\psi^* = \partial_t\psi^* = 0\,.$$

Since we have shown $\psi^*(x,0) = \phi$ for $x < x_-(\eta)$ we conclude

$$(5.40)\qquad \int_{x_-(\eta,t)}^{x_+(\eta,t)} \frac{P(\eta)}{R(\eta)}\,dy = \phi(\eta).$$

Finally, since we have shown above that $\psi^*(x,t)$ given by (5.39) is

a positive, decreasing function of x, it follows from (5.40) that $0 \leq \psi^* \leq \phi$, i.e. that ψ^* belongs to the admissible set A.

The main result of this section is

Theorem 5.11. The solution of the variational problem (2.16-2.18) is given by

$$(5.41)\quad \psi^*(x,t) = \begin{cases} 0 & \text{for} \quad x \geq x_+(\eta,t) \\ \int_x^{x_+(\eta,t)} \frac{P(\eta)}{R(\eta)}\,dy & \text{for } x_-(\eta,t) < x < x_+(\eta,t) \\ \phi(\eta) & \text{for} \quad x \leq x_-(\eta,t)\ . \end{cases}$$

Proof: We will show that $\psi^*(x,t)$ given above satisfies the variational conditions (3.34). Introducing

$$(5.42)\qquad \theta_+(\eta,t) = \eta\, x_+(\eta,t) - L\psi^*\big(x_+(\eta,t),t\big),$$

we integrate (5.18) with respect to x to obtain

$$(5.43)\quad L\psi^*(x,t) = \begin{cases} \eta x - \theta_+(\eta,t) - \int_{x_+(\eta,t)}^{x} \int_0^{\eta} \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\eta\ dy & \text{for } x > x_+(\eta,t) \\ \eta x - \theta_+(\eta,t) & \text{for } x_-(\eta,t) \leq x \leq x_+(\eta,t) \\ \eta x - \theta_+(\eta,t) + \int_x^{x_-(\eta,t)} \int_0^{\eta} \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\mu\ dy & \text{for } x < x_-(\eta,t). \end{cases}$$

Differentiating (5.42) and using (3.50) shows

$$(5.44)\qquad \partial_t\theta_+ = \eta\partial_t x_+ - L\psi_x^*(x_+,t)\ \partial_t x_+ - L\psi_t^*(x_+,t) = 4\eta^3\ .$$

We know from (4.16) that

$$\theta_+(\eta,0) = \eta\, x_+(\eta,0) - L\psi^*(x_+(\eta,0),0)$$
$$= \eta x_+(\eta) - L\psi^*(x_+(\eta),0) = \theta_+(\eta).$$

Therefore integrating (5.44) we see that

$$\theta_+(\eta,t) = 4\eta^3 t + \theta_+(\eta) \tag{5.45}$$

and, using the definition (1.23) of a, that

$$a(\eta,x,t) = \eta x - \theta_+(\eta,t). \tag{5.46}$$

Thus, using (5.46), formula (5.43) becomes

$$a - L\psi^* = \begin{cases} \displaystyle\int_{x_+(\eta,t)}^{x} \int_0^{\eta} \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\eta \; dy & \text{for } x > x_+(\eta,t) \\ 0 & \text{for } x_-(\eta,t) \le x \le x_+ \\ \displaystyle -\int_{x}^{x_-(\eta,t)} \int_0^{\eta} \mathrm{Im}\left(\frac{P(\mu)}{R(\mu)}\right) d\mu \; dy & \text{for } x < x_-(\eta,t). \end{cases} \tag{5.47}$$

Using (5.38) of Lemma 5.10 we conclude from this that $\psi^*(x,t)$ given by (5.41) satisfies the variational conditions (3.34). This proves Theorem 5.11.

We may obtain explicit formulas for $\overline{u}(x,t)$, $\overline{u^2}(x,t)$ and $\overline{u^3};(x,t)$ by substituting ψ_x^* given by (5.16) and ψ_t^* given by (5.17) into (3.59) of Theorem 3.17:

(5.48) $$\overline{u} = \frac{4}{\pi}\,(\eta,\psi_x^*) = -\frac{2}{\pi}\int_I \eta\,\frac{P(\eta)}{R(\eta)}\,d\eta$$

(5.49) $$\overline{u^2} = -\frac{16}{3\pi}\,(\eta^3,\psi_x^*) = \frac{8}{3\pi}\int_I \eta^3\,\frac{P(\eta)}{R(\eta)}\,d\eta$$

(5.50) $$= \frac{4}{3\pi}\,(\eta,\psi_t^*) = \frac{2}{3\pi}\int_I \eta\,\frac{Q(\eta)}{R(\eta)}\,d\eta$$

(5.51) $$\overline{u^3} = -\frac{4}{3\pi}\,(\eta^3,\psi_t^*) = -\frac{2}{3\pi}\int_I \eta^3\,\frac{Q(\eta)}{R(\eta)}\,d\eta\ .$$

These integrals can be evaluated by contour integration. Slit the plane along the intervals of the set I given by (5.1); the integral along each interval is then equal to half the integral of a clockwise contour around its slit. By Cauchy's integral theorem we can deform the contour to a large clockwise circle around the origin containing I. Then letting the radius of this circle go to infinity we can evaluate the integral by computing its residue at infinity.

For large $|\zeta|$ we have by (5.4)

$$R^2(\zeta) = \prod_{k=1}^{n} (\beta_k^2 - \zeta^2)$$

(5.52) $$= (-\zeta^2)^n \prod_{k=1}^{n} \Big(1 - \frac{\beta_k^2}{\zeta^2}\Big)$$

$$= (-\zeta^2)^n\Big[1 - \Big(\sum_k \beta_k^2\Big)\frac{1}{\zeta^2} + \Big(\sum_{ij} \beta_i^2\beta_j^2\Big)\frac{1}{\zeta^4} - \Big(\sum_{ijk} \beta_i^2\beta_j^2\beta_k^2\Big)\frac{1}{\zeta^6} + O\Big(\frac{1}{|\zeta|^8}\Big)\Big]$$

where the sums over more than one index are understood never to repeat an index value in any term. Since

$$(1+\omega)^{-1/2} = 1 - \frac{1}{2}\omega + \frac{3}{8}\omega^2 - \frac{5}{16}\omega^3 + O(\omega^4),$$

we see from (5.52) and (5.5) that

$$\frac{1}{R(\zeta)} = \frac{i}{\zeta^n}\Big[1 + \frac{1}{2}\Big(\sum_k \beta_k^2\Big) \frac{1}{\zeta^2} + \Big(\frac{3}{8}\Big(\sum_k \beta_k^2\Big)^2 - \frac{1}{2}\Big(\sum_{ij} \beta_i^2\beta_j^2\Big)\Big) \frac{1}{\zeta^4}$$

(5.53)

$$+ \Big(\frac{5}{16}\Big(\sum_k \beta_k^2\Big)^3 - \frac{3}{4}\Big(\sum_k \beta_k^2\Big) \Big(\sum_{ij} \beta_i^2\beta_j^2\Big) + \frac{1}{2}\Big(\sum_{ijk} \beta_i^2\beta_j^2\beta_k^2\Big)\Big) \frac{1}{\zeta^6} + O\Big(\frac{1}{|\zeta|^8}\Big) \Big] .$$

From (5.8) and (5.9) we see that

$$P(\zeta) = \zeta^n\Big[1 + \frac{\alpha_2}{\zeta^2} + \frac{\alpha_4}{\zeta^4} + O\Big(\frac{1}{|\zeta|^6}\Big)\Big] ,$$

$$Q(\zeta) = \zeta^{n+2}\Big[12 - 6\Big(\sum_k \beta_k^2\Big) \frac{1}{\zeta^2} + \frac{\gamma_2}{\zeta^4} + \frac{\gamma_4}{\zeta^6} + O\Big(\frac{1}{|\zeta|^8}\Big)\Big].$$

Combining this with (5.53) gives

$$\frac{P(\zeta)}{R(\zeta)} = i\Big[1 + \Big(\alpha_2 + \frac{1}{2}\Big(\sum_k \beta_k^2\Big) \frac{1}{\zeta^2}$$

(5.54)

$$+ \Big(\alpha_4 + \frac{1}{2}\Big(\sum_k \beta_k^2\Big)\alpha_2 + \frac{3}{8}\Big(\sum_k \beta_k^2\Big)^2 - \frac{1}{2}\Big(\sum_{ij} \beta_i^2\beta_j^2\Big)\Big) \frac{1}{\zeta^4} + O\Big(\frac{1}{|\zeta|^6}\Big)\Big] ,$$

$$\frac{Q(\zeta)}{R(\zeta)} = i\Big[12\,\zeta^2 + \Big(\gamma_2 + \frac{3}{2}\Big(\sum_k \beta_k^2\Big)^2 - 6\Big(\sum_{ij} \beta_i^2\beta_j^2\Big)\Big) \frac{1}{\zeta^2}$$

$$(5.55) \qquad + \left(\gamma_4 + \frac{1}{2}\left(\sum_k \beta_k^2\right)\gamma_2 + \frac{3}{2}\left(\sum_k \beta_k^2\right)^3\right.$$

$$\left. - 6\left(\sum_k \beta_k^2\right)\left(\sum_{ij} \beta_i^2\beta_j^2\right) + 6\left(\sum_{ijk} \beta_i^2\beta_j^2\beta_k^2\right)\right)\frac{1}{\zeta^4} + O\left(\frac{1}{|\zeta|^6}\right) \Big] .$$

Using (5.54) and (5.55) to find the residues, we compute from (5.48-5.51):

$$(5.56) \quad \overline{u} = \frac{1}{\pi}\oint \zeta \frac{P(\zeta)}{R(\zeta)}\,d\zeta = -\left(\sum_k \beta_k^2\right) - 2\alpha_2$$

$$(5.57) \quad \overline{u^2} = -\frac{4}{3\pi}\oint \zeta^3 \frac{P(\zeta)}{R(\zeta)}\,d\zeta = \left(\sum_k \beta_k^2\right)^2 - \frac{4}{3}\left(\sum_{ij} \beta_i^2\beta_j^2\right) + \frac{4}{3}\left(\sum_k \beta_k^2\right)\alpha_2 + \frac{8}{3}\alpha_4$$

$$(5.58) \quad \overline{u^2} = -\frac{1}{3\pi}\oint \zeta \frac{Q(\zeta)}{R(\zeta)}\,d\zeta = \left(\sum_k \beta_k^2\right)^2 - 4\left(\sum_{ij} \beta_i^2\beta_j^2\right) + \frac{2}{3}\gamma_2$$

$$(5.59) \quad \overline{u^3} = \frac{1}{3\pi}\oint \zeta^3 \frac{Q(\zeta)}{R(\zeta)}\,d\zeta = -\left(\sum_k \beta_k^2\right)^3 + 4\left(\sum \beta_k^2\right)\left(\sum_{ij} \beta_i^2\beta_j^2\right)$$

$$- 4\left(\sum_{ijk} \beta_i^2\beta_j^2\beta_k^2\right) - \frac{1}{3}\left(\sum_k \beta_k^2\right)\gamma_2 - \frac{2}{3}\gamma_4 .$$

Note that together, (5.57) and (5.58) yield an algebraic identity

$$\gamma_2 = 4\sum_{ij} \beta_i^2\beta_j^2 + 2\left(\sum_k \beta_k^2\right)\alpha_2 + 4\alpha_4 .$$

Now we consider the region of x and t where n, the cardinality of B(x,t) is equal to one. In that case formulas (5.56 - 5.59) become

$$(5.60) \qquad \overline{u} = -\beta^2 , \quad \overline{u^2} = \beta^4 , \quad \overline{u^3} = -\beta^6 ,$$

which are (4.39) and (4.41).

6. Asymptotic Behavior for Large Time

The main result of this section concerns the values of $\bar{u}(x,t)$ as $t \to \infty$:

Theorem 6.1. Let δ be any positive quantity:

(a) For $x/t < -\delta$

$$\bar{u}(x,t) = O(t^{-2}). \tag{6.1}$$

(b) For $\delta < x/5 < 4-\delta$

$$\bar{u}(x,t) = -\frac{1}{2\pi t}\,\phi(\sigma(x,t)) + o(t^{-1}), \tag{6.2}$$

where

$$\sigma(x,t) = \frac{1}{2}(x/t)^{1/2}. \tag{6.2$'$}$$

(c) For $x/t > 4 + \delta$

$$\bar{u}(x,t) = O(t^{-2}). \tag{6.3}$$

The proof is based on the variational characterization of $\bar{u}$. We use condition (3.35):

$$\psi^*(\eta,x,t) = \begin{cases} 0 & \text{where } \eta x - 4\eta^3 t - \theta_+(\eta) > 0 \\ \phi(\eta) & \text{where } \eta x - 4\eta^3 t - \theta_-(\eta) < 0. \end{cases} \tag{6.4}$$

According to Lemma 1.4, $\theta_+(\eta)$ and $\theta_-(\eta)$ are continuous functions that vanish at $\eta = 0$. From this and (6.4) we deduce

Lemma 6.2. (a) For $x/t < -\delta$,

$$\psi^*(\eta) = \phi(\eta) \quad \text{for} \quad O(t^{-1}) < \eta < 1 . \tag{6.5}$$

(b) For $\delta < x/t < 4-\delta$

$$\psi^*(\eta) = \begin{cases} 0 & \text{for} \quad O(t^{-1}) < \eta < \sigma(x,t) - O(t^{-1}) \\ \\ \phi(\eta) & \text{for} \quad \sigma(x,t) + O(t^{-1}) < \eta < 1. \end{cases} \tag{6.6}$$

(c) For $x/t > 4+\delta$

$$\psi^*(\eta) = 0 \quad \text{for} \quad O(t^{-1}) < \eta \leq 1. \tag{6.7}$$

We show now that parts (a) and (c) of Theorem 6.1 follow from parts (a) and (c) of Lemma 6.2. It follows from (6.5) and (6.7) that in both cases the parameters β_j introduced in Section 4 lie in the range $(0, O(t^{-1}))$. We assume that there is in fact only a single β.

According to formula (5.60)

$$\overline{u} = - \beta_1^2 ;$$

since $\beta_1 \leq O(t^{-1})$, (6.1) and (6.3) follow.

We turn now to the interesting case (b); here it follows from (6.6) that the parameters β_j are contained in the intervals

$$[0, O(t^{-1})] \quad \text{or} \quad [\sigma(x,t) - O(t^{-1}), \sigma(x,t) + O(t^{-1})]$$

with at least two β's in the second interval. We assume that in fact there are exactly two, β_1 and β_2, in the second interval, and one, β_3, in the first. Thus (6.6) can be rewritten as

$$\psi^*(\eta) = \begin{cases} 0 & \text{for } \eta \text{ in } [\beta_3, \beta_2] \\ \phi(\eta) & \text{for } \eta \text{ in } [\beta_1, 1] \\ \text{in } (0, \phi(\eta)) & \text{for } \eta \text{ in } (0, \beta_3) \text{ and } (\beta_2, \beta_1). \end{cases} \tag{6.8}$$

Here

$$|\beta_j - \sigma(x,t)| \le O(t^{-1}), \quad j = 1,2 \tag{6.9}$$

and

$$\beta_3 = O(t^{-1}). \tag{6.9$'$}$$

The next lemma nails down more precisely the distance between β_1 and β_2. We assume that ϕ is continuous in $(0,1)$ and that θ_+ is differentiable.

<u>Lemma 6.3.</u>

$$\lim_{t\to\infty} \frac{\log(\beta_1 - \beta_2)}{x} \phi(\sigma(x,t)) = -2\pi . \tag{6.10}$$

<u>Proof</u>: According to the variational condition (3.34),

$$L\psi^* = a \tag{6.11}$$

at every point η where $0 < \psi^*(\eta) < \phi(\eta)$. In particular, according to (6.8), (6.11) holds for η in (β_2,β_1), and so by continuity also at $\eta = \beta_i$, $i = 1,2$. We now set $\eta = \beta_i$ in (6.11); using the definition (2.20) of L we get

$$(6.11)' \quad a(\beta_i) = (L\psi^*)(\beta_i) = \frac{1}{2\pi}\int_0^1 \log\left(\frac{\beta_i-\mu}{\beta_i+\mu}\right)^2 \psi^*(\mu)\, d\mu\ , \quad i = 1,2.$$

According to (6.8), $\psi^*(\mu) = 0$ for $\beta_3 \leq \mu \leq \beta_2$ and $\psi^*(\mu) = \phi(\mu)$ for $\beta_1 \leq \mu \leq 1$. Using this information, we obtain the following:

$$(6.12) \qquad a(\beta_1) - a(\beta_2) = (L\psi^*)(\beta_1) - (L\psi^*)(\beta_2)$$

$$= \frac{1}{2\pi}\int_0^{\beta_3} \log\left(\frac{(\beta_1-\mu)(\beta_2+\mu)}{(\beta_2-\mu)(\beta_1+\mu)}\right)^2 \psi^*(\mu)\, d\mu$$

$$+ \frac{1}{2\pi}\int_{\beta_2}^{1} \log\left(\frac{\beta_2+\mu}{\beta_1+\mu}\right)^2 \psi^*(\mu)\, d\mu$$

$$+ \frac{1}{2\pi}\int_{\beta_2}^{\beta_1} \log\left(\frac{\beta_1-\mu}{\beta_2-\mu}\right)^2 \psi^*(\mu)\, d\mu$$

$$+ \frac{1}{2\pi}\int_{\beta_1}^{1} \log\left(\frac{\beta_1-\mu}{\beta_2-\mu}\right)^2 \phi(\mu)\, d\mu\ .$$

We claim that the first three terms on the right are $O(\beta_1-\beta_2)$. This is obvious for the first two terms; in the third term we introduce as new variable of integration

$$\lambda = \frac{\mu - \beta_2}{\beta_1 - \beta_2}\ .$$

We get

$$(\beta_1-\beta_2)\int_0^1 \log\left(\frac{1-\lambda}{\lambda}\right)\psi^* \, d\lambda \ .$$

Since $\psi^* \leq \phi \leq$ const., the above quantity is indeed $O(\beta_1 - \beta_2)$. Thus we can rewrite (6.12) as

$$(6.13) \qquad a(\beta_1) - a(\beta_2) = \frac{1}{2\pi}\int_{\beta_1}^{1} \log\left(\frac{\beta_1-\mu}{\beta_2-\mu}\right)^2 \phi(\mu)\, d\mu + O(\beta_1-\beta_2) \ .$$

We further split the integral on the right as

$$\int_{\beta_1}^{1} = \int_{\beta_1}^{\beta_1+\gamma} + \int_{\beta_1+\gamma}^{1} ,$$

γ a small quantity. For fixed γ, the second integral is $O(\beta_1-\beta_2)$; in the first integral we write

$$\phi(\mu) = \phi(\beta_1) + \varepsilon(\mu) ;$$

$\varepsilon(\mu)$ is small in $[\beta_1,\beta_1+\gamma]$ when γ is small. We introduce

$$\kappa = \mu - \beta_1$$

as new variable of integration; this allows us to rewrite the right side of (6.13) as

$$(6.13)' \qquad \frac{1}{\pi}\int_0^{\gamma} [\phi(\beta_1)+\varepsilon] \log \frac{\kappa}{\kappa+(\beta_1-\beta_2)}\, d\kappa + O(\beta_1-\beta_2)$$

$$= \frac{1}{\pi} [\phi(\beta_1)+\varepsilon] (\beta_1-\beta_2) \log (\beta_1-\beta_2) + O(\beta_1-\beta_2) \ .$$

We turn now to the left side of (6.13) and apply the mean value theorem; we get

(6.14) $$a(\beta_1) - a(\beta_2) = \overline{a_\eta}(\beta_1 - \beta_2)$$

where

(6.15) $$\overline{a_\eta} = a_\eta(\bar\beta), \quad \beta_2 < \bar\beta < \beta_1 .$$

It follows from (6.9) that

(6.16) $$\bar\beta = \sigma(x,t) + O(t^{-1}) .$$

We use now definition (1.23) of $a(\eta,x,t)$ to write

$$a_\eta = x - 12t\eta^2 - d\theta_+/d\eta .$$

using this and (6.16) in (6.15) we get

$$\overline{a_\eta} = x - 12tx/4t + O(1) = -2x + O(1) .$$

Using this in (6.14) we get

$$a(\beta_1) - a(\beta_2) = -2x(\beta_1 - \beta_2) + O(\beta_1 - \beta_2) .$$

This is equal to (6.13)′; dividing by $(\beta_1 - \beta_2)$ we get

$$-2x + O(1) = \frac{1}{\pi}(\phi(\beta_1) + \varepsilon)\log(\beta_1 - \beta_2) + O(1).$$

Using (6.9) once more we note that

$$\phi(\beta_1) = \phi(\sigma(x,t)) + \varepsilon(t) ,$$

where $\varepsilon(t) \to 0$ as $t \to \infty$; relation (6.10) of Lemma 6.3 follows.

We make use of formula (5.56) for the limit solution:

$$\bar{u} = -(\beta_1^2+\beta_2^2+\beta_3^2) - 2\alpha_2 ; \tag{6.17}$$

α_2 is a function of β_1,β_2,β_3 defined through (5.20). With $n = 3$,

$$P(z) = z^3 + \alpha_2 a$$

so (5.20) is

$$\int_{\beta_3}^{\beta_2} \frac{\eta^3 + \alpha_2\eta}{R(\eta)} d\eta = 0 .$$

Thus

$$\alpha_2 = - J^{(3)} / J^{(1)} , \tag{6.18}$$

where

$$J^{(j)} = \int_{\beta_3}^{\beta_2} \frac{\eta^j}{R(\eta)} d\eta , \quad j = 1,3, \tag{6.19}$$

and

$$R^2(\eta) = (\eta^2-\beta_3^2)(\beta_2^2-\eta^2)(\beta_1^2-\eta^2) . \tag{6.20}$$

According to (6.9), (6.9)′, as $t \to 0$, $\beta_1-\beta_2 \to 0$ and $\beta_3 \to 0$. We

shall now determine the asymptotic behavior of α_2 under these circumstances.

Lemma 6.4. As $\beta_1 - \beta_2$ and $\beta_3 \to 0$,

(a)

$$\beta_1^2 J^{(1)} - J^{(3)} - \beta_1 \to 0 \tag{6.21}$$

(b)

$$J^{(1)} \simeq \frac{1}{2\beta_1} |\log (\beta_1 - \beta_2)| . \tag{6.21'}$$

Proof: (a) Using (6.19) and (6.20) we get

$$\beta_1^2 J^{(1)} - J^{(3)} = \int_{\beta_3}^{\beta_2} \frac{\beta_1^2 \eta - \eta^3}{R(\eta)} d\eta \tag{6.22}$$

$$= \int_{\beta_3}^{\beta_2} \frac{\eta}{(\eta^2 - \beta_3)^{1/2}} \left(\frac{\beta_1^2 - \eta^2}{\beta_2^2 - \eta^2}\right)^{1/2} d\eta .$$

The integrand on the right in (6.22) tends to 1 uniformly on every closed subset of (β_3, β_2), and is dominaed over that interval by const./$[(\eta - \beta_3)(\beta_2 - \eta)]^{1/2}$, an integrable function. Therfore by the principle of dominated convergence as $\beta_3 \to 0$ and $\beta_2 - \beta_1 \to 0$, (6.22) approximates β_1, as asserted in (6.21).

To prove (6.21)′ we separate those factors of R which are small at the end point β_1 :

$$R(\eta) = [(\eta^2-\beta_3^2)(\beta_2+\eta)(\beta_1+\eta)]^{1/2}[(\beta_1-\eta)(\beta_2-\eta)]^{1/2}.$$

The value of the first factor at $\eta = \beta_2$ is

$$2\beta_1^2 + O(\beta_1 - \beta_2) + O(\beta_3) .$$

From this we deduce easily that

$$(6.22)' \qquad J^{(1)} \simeq \frac{1}{2\beta_1} \int_{\beta_3}^{\beta_2} \frac{d\eta}{[(\beta_1-\eta)(\beta_2-\eta)]^{1/2}} ;$$

elementary calculus shows that the integral on the right in (6.22)′ is $\simeq -\log(\beta_1-\beta_2)$; this completes the proof of part (b) of Lemma 6.4.

Using (6.21), (6.21)′ in (6.18) gives

$$\alpha_2 = -J^{(3)}/J^{(1)} = \frac{\beta_1^2 J^{(1)}-J^{(3)}}{J^{(1)}} - \beta_1^2$$

$$= -\beta_1^2 + \frac{2\beta_1^2}{\log(\beta_1-\beta_2)} + o\left(\frac{1}{\log(\beta_1-\beta_2)}\right) .$$

Setting this into (6.17) and using (6.10) we get

$$(6.23) \qquad \overline{u}(x,t) = \beta_1^2 - \beta_2^2 - \beta_3^2 - \frac{2\beta_1^2}{\pi x}\,\phi(\sigma(x,t)) + o(t^{-1}).$$

It follows from (6.20) that

$$|\beta_1 - \beta_2| \leq O(e^{-c/t}) .$$

Using this to estimate the first two terms, (6.9)′ to estimate the third term and (6.9) to estimate the fourth term we obtain (6.2). This completes the proof of Theorem 6.1.

We conclude with a number of observations:

(i) A similar argument, baed on formula (5.57) for $\overline{u^2}$, yields

$$\overline{u^2}(x,t) = \frac{x}{6\pi t^2}\,\phi(\sigma(x,t)) + o(t^{-1}). \tag{6.24}$$

The first relation of (4.42)is

$$\overline{u}_t = 3\overline{u^2}_x \ ; \tag{6.25}$$

the leading terms in the asymptotic descriptions (6.2) and (6.24) are consistent with this relation.

(ii) Relation (6.6) suggests the following crude approximation to $\psi^*(\eta,x,t)$ in case (b):

$$\psi^*(\eta,x,t) = H(\eta - \sigma(x,t))\,\phi(\eta) \ ,$$

where

$$H(\mu) = 0 \ \text{ for } \mu < 0 \ , \ = 1 \text{ for } \mu > 0 \ .$$

Then, using (6.2)′,

$$\psi^*_x(\eta,x,t) = -\,\delta(\eta - \sigma(x,t))\,\frac{1}{4(xt)^{1/2}}\,\phi(\eta) \ .$$

Using this expression in formula (3.59) for $\bar{u}$ gives

$$\bar{u}(x,t) = \frac{4}{\pi}(\eta,\psi_x^*) = -\frac{4}{\pi}\sigma(x,t)\frac{\phi(\sigma(x,t))}{4(xt)^{1/2}}$$

$$= -\frac{1}{2\pi t}\phi(\sigma(x,t)) ,$$

in agreement with the leading term in (6.2).

(iii) Next we show that (6.2) can be derived from the zero dispersion limit of Tanaka's description of the large t behavior of solutions of KdV. Tanaka shows that for large t, and for $0 < x < 4t$ every solution is a wave train, i.e. approximately a superposition of solitons

(6.26) $$u(x,t;\varepsilon) \geqslant \sum_1^N s(x - 4\eta_n^2 t - \delta_n , \eta_n) ;$$

here

(6.26)′ $$s(x,\eta) = -2\eta^2 \operatorname{sech}^2(\eta x/\varepsilon) .$$

The width of such a solition is

(6.27) $$\text{const. } \varepsilon/\eta .$$

It follows from the precise form $(1.18)_{ii}$ of Weyl's law that

(6.28) $$\eta_{n+1} - \eta_n \simeq \frac{\pi\varepsilon}{\phi(\bar{\eta}_n)} , \quad \bar{\eta}_n \text{ in } [\eta_n,\eta_{n+1}] .$$

Peaks of the wave train (6.26) are located at

$$4\eta_n^2 t + \delta_n \; . \tag{6.29}$$

Using (6.28) we see that for t large the peaks are separated by the distance

$$\simeq \; 8\eta_n t(\eta_{n+1} - \eta_n) + \delta_{n+1} - \delta_n \simeq \frac{8\pi\eta_n t\varepsilon}{\phi(\eta_n)} \; . \tag{6.29)'$$

Comparing this to (6.27) we conclude that for large t the individual solitons in the wave train are well separated.

The wave number η of the soliton that peaks at x at time t is by (6.29) and (6.2)′,

$$\eta_n \simeq \sigma(x,t) = (x/4t)^{1/2},$$

provided that t is large and $0 < x/t < 4$. Setting this in (6.29)′ we conclude that for t large the density of solitions at x is

$$\simeq \phi(\sigma(x,t))/4\pi\varepsilon(xt)^{1/2} \; . \tag{6.30}$$

It follows from (6.26)′ that the area under a soliton is $4\eta\varepsilon$; at the wave number (6.29) this is

$$2(x/t)^{1/2}\,\varepsilon \; . \tag{6.30)'$$

The asymptotic area density in the wave tain (6.26) is the product of (6.30) and (6.30)′:

$$(2\pi t)^{-1} \phi(\sigma(x,t)) .$$

Since the asymptotic area density is the weak limit $\bar{u}$, we obtain yet another derivation of (6.2).

(iv) In deriving (6.2) we have assumed that the function ϕ is continuous. When the potential has several local minima, the function ϕ has discontinuities; in these cases Theorem 6.1 remains true as long as $\sigma(x,t)$ is bounded away from the points of discontinuity of ϕ.

7. Initial data with $u(-\infty) \neq u(\infty)$.

In this section we study solutions of

$$u_t - 6uu_x + \varepsilon^2 u_{xxx} = 0 \tag{7.1}$$

whose initial value u satisfies

$$\lim_{x\to-\infty} u(x) = -1, \quad \lim_{x\to\infty} u(x) = 0, \tag{7.2}$$

and is an increasing function of x.

The solution of such an initial value problem can be obtained as the limit of solutions whose initial data u_n satisfy the conditions laid down in Section 1:

$$\lim_{n\to\infty} u_n(x) = u(x) \ , \tag{7.3}$$

uniformly for $x > x_0$, x_0 any number. Denoting by $u_n(x,t,\varepsilon)$ the solution of (7.1) with initial data $u_n(x)$, it can be shown that

$$\lim_{n\to\infty} u_n(x,t;\varepsilon) = u(x,t;\varepsilon) \tag{7.3)'}$$

Although for the solutions of KdV equations signals propagate with infinite speed, the dependence of the solution at fixed x,t on the values of the initial data at y diminishes to zero as y tends to ∞, uniformly in ε. Therefore we conjecture tha the limits $n \to \infty$ and $\varepsilon \to 0$ can be interchanged:

Conjecture 7.1:

$$(7.4)\qquad \underset{\varepsilon\to 0}{\text{d-lim}}\; u(x,t,\varepsilon) = \lim_{n\to\infty} \underset{\varepsilon\to 0}{\text{d-lim}}\; u_n(x,t,\varepsilon).$$

The inner limit on the right in (7.5) is characterized in Theorem 2.10 in terms of the solution of a minimum problem (2.16): :

$$(7.5)\qquad \overline{u}_n = \text{d-lim}\, u_n(x,t,\varepsilon) = \partial_x^2 Q_n^* ,$$

where

$$(7.5)'\qquad Q_n^* = \min_{\phi\le\psi\le\phi_n} Q_n(\psi;x,t),$$

and

$$(7.6)\qquad , Q_n(\psi) = \frac{4}{\pi}(a_n,\psi) - \frac{2}{\pi}(L\psi,\psi) .$$

By (1.23)

$$(7.7)\qquad a_n = x\eta - 4k\eta^3 - \theta_+^{(n)}(\eta) ;$$

by (1.25)

$$(7.8)\qquad \phi_n(\eta) = \int_{x_-^{(n)}(\eta)}^{x_+^{(n)}(\eta)} \frac{\eta}{|u_n(y)+\eta^2|^{1/2}}\, dy ,$$

and by (1.16)

$$(7.9)\qquad \theta^{(n)}(\eta) = \eta x_+^{(n)}(\eta) + \int_{x_+^{(n)}}^{\infty} \left(\eta - (u_n(y) + \eta^2\right)^{1/2} dy .$$

$x_\pm^{(n)}(\eta)$ are defined by (1.):

$$(7.10) \qquad u_n(x_{\pm}^{(n)}(\eta)) = -\eta^2 .$$

Since by (7.3), u_n tend to u, and u satisfies (7.2), it follows that

$$(7.11) \qquad \lim_{n\to\infty} x_{-}^{(n)}(\eta) = -\infty , \quad \lim_{n\to\infty} x_{+}^{(n)}(\eta) = x_{+}(\eta)$$

From this we deduce easily that

$$\lim_{n\to\infty} \theta_{+}^{(n)}(\eta) = \theta_{+}(\eta) ,$$

$$\lim_{n\to\infty} a_n(x,t,\eta) = a(x,t,\eta) ,$$

$$\lim_{n\to\infty} \phi_n(\eta) = \infty ,$$

uniformly in η. It follows from this that for fixed ψ in L^1,

$$(7.11) \qquad \lim_{n\to\infty} Q_n(\psi;x,t) = Q(\psi,x,t)$$

We surmise tha the limit of the minima Q_n^* defined in (7.5)′ can be characterized as the minimum of (7.11):

<u>Conjecture 7.2:</u>

$$(7.12) \qquad \lim Q_n^* = Q^*$$

where

$$(7.13) \qquad Q^* = \min_{0\leq\psi} Q(\psi) ,$$

$$(7.13)' \qquad Q(\psi) = \frac{4}{\pi}(a,\psi) - \frac{2}{\pi}(L\psi,\psi) .$$

Combining conjectures 7.1 and 7.2 we deduce

Conjecture 7.3:

$$\text{(7.14)} \qquad \underset{\varepsilon \to 0}{\text{d-lim}} \quad u(x,t,\varepsilon) = \partial_x^2 Q^* ,$$

where Q^* is defined by (7.13).

It is far from obvious that the minimum problem (7.13) has a solution. Nevertheless it follows from the positive definiteness of Q that it ψ^* satisfies the variational conditions for the minimum problem, then $Q(\psi)$ achieves its minimum at ψ^*, and only at ψ^*. The variational conditions (3.34) in the present case are:

$$\text{(7.15)} \qquad \psi^* = 0 \quad \text{where} \quad a - L\psi^* > 0$$

$$\text{(7.15)}' \qquad a - L\psi^* \geq 0 \quad \text{for all } \eta .$$

We turn now to the special initial function

$$\text{(7.16)} \qquad u(x) = \begin{cases} -1 & \text{for} \quad x < 0 \\ 0 & \text{for} \quad x > 0 \end{cases}$$

In his case $x_+(\eta) \equiv 0$, $\theta_+(\eta) \equiv 0$, and

$$\text{(7.17)} \qquad a = x\eta - 4t\eta^3 .$$

Equation (7.13) can be solved explicitly:

Theorem 7.4: For a given by (7.17), the solution ψ^* of the variational problem (7.13) is as follows:

(a) For

$$\text{(7.18)} \qquad x < -6t$$

(7.18)′ $$\psi^* = \frac{12\eta^3-(6t+x)\eta}{(1-\eta^2)^{1/2}} .$$

(b) For

(7.19) $$-6t < x < 4t$$

(7.19)′ $$\psi^* = \begin{cases} 0 \text{ for } \eta < \beta \\ \dfrac{S}{R_0} \quad \text{for } \beta < \eta < 1 \end{cases}$$

(7.20) $$R_0^2(\eta) = (1-\eta^2)(\eta^2-\beta^2) ,$$

(7.21) $$S(\eta,x,t) = 12tQ - xP ,$$

(7.21)′ $$Q(\eta) = \eta^4 + \gamma_1\eta^2 + \gamma_2 , \quad P(\eta) = \eta^2 + \alpha .$$

where

(7.22) $$\alpha = - I^{(2)}/I^{(0)}$$

(7.23) $$\gamma_1 = \frac{1 + \beta^2}{2}$$

(7.24) $$\gamma_2 = - (\gamma_1 I^{(2)} + I^{(4)} / I^{(0)} .$$

$I^{(j)}$ denotes the complete elliptic integal

(7.25) $$I^{(j)} = \int_0^\beta \frac{\eta^j}{R(\eta)} d\eta ,$$

(7.25)′ $$R^2(\eta) = (1-\eta^2)(\beta^2-\eta^2) .$$

The parameter β is related to x/t through

(7.26) $$S(\beta,x,t) = 0 .$$

(c) For

(7.27) $$4t < x ,$$

(7.27)′ $$\psi^* \equiv 0$$

Proof: We verify (c) first. Clearly ψ^* is nonnegative and so admissible. We show now that the variational conditions are satisfied. Since $\psi^* = 0$, $L\psi^* = 0$; combining this with (7.17) we obtain

(7.28) $$a - L\psi^* = x\eta - 4t\eta^3 .$$

Clearly, if (7.27) holds, (7.28) is ≥ 0 in the range $0 \leq \eta \leq 1$; thus (7.15) and (7.15)′ are satisfied.

Next we take case (a). It follows from (5.16), (5.17), $n = 1$, that for ψ^* defined by (7.18)′,

$$a - L\psi^* = 0 \quad \text{for} \quad 0 \leq \eta \leq 1.$$

This shows that the variational conditions are satisfied. It further follows from (7.18) that ψ^* as defined by (7.18)′ is ≥ 0 for $\eta \geq 0$ and therefore is admissible.

We turn now to case (b). Fix β in $[0,1]$, and define x/t through the relation (7.26).

We can write ψ^*, given by (7.19)′, as

(7.29) $$\psi^* = x\psi_1^* - 12t\psi_2^*$$

where

$$\psi_1^* = \begin{cases} 0 & \text{for } \eta < \beta\,, \\ \dfrac{P}{R_0}\,, \quad P = \eta^2 + \alpha & \text{for } \beta < \eta < 1 \end{cases}$$

(7.30)

$$\psi_2^* = \begin{cases} 0\,, & \text{for } \eta < \beta\,, \\ \dfrac{Q}{R_0}\,, \quad Q = \eta^4 + \gamma_1\eta^2 + \gamma_2 & \text{for } \beta < \eta < 1\,, \end{cases}$$

see formulas (5.16), (5.17), $n = 2$.

We see that ψ_1^* and ψ_2^* are real parts on the real axis of the analytic functions, see (5.16), (5.17), $n = 2$:

$$f_1(\eta) = \frac{P(\eta)}{R_0(\eta)} - i\,,$$

(7.31)

$$f_2(\eta) = \frac{Q(\eta)}{R_0(\eta)} - i\eta^2\,.$$

Clearly, f_1 is bounded in the upper half plane; using (7.23) we see that so is f_2.

Using (7.29), (7.31) and (7.21) we get

$$\psi^* = \operatorname{Re} f\,, \tag{7.32}$$

$$f(\eta) = \frac{S(\eta)}{R_0(\eta)} + i(x-12t\eta^2)\,. \tag{7.32$'$}$$

Using relation (3.41) between the operator L and the Hilbert transform H we deduce from (7.32), (7.32)′ that

$$L\psi^* = \int_0^\eta H\,\psi^*\,d\mu = \int_0^\eta \mathrm{Im}(f)\,d\mu$$

(7.33)
$$= \begin{cases} -\displaystyle\int_0^\eta \frac{S(\mu)}{R(\mu)}\,d\mu + x\eta - 4\eta^3\,, & \eta < \beta \\[2ex] -\displaystyle\int_0^\beta \frac{S(\mu)}{R(\mu)}\,d\mu + x\eta - 4\eta^3\,, & \eta \geqslant \beta \end{cases}$$

where R , defined by (7.25)′, is positive on $(0,\beta)$.

Relations (7.22) and (7.24) guarantee that

(7.34)
$$\int_0^\beta \frac{S(\mu)}{R(\mu)}\,d\mu = 0\,.$$

Combining this with (7.33) and using (7.17) we obtain

(7.35)
$$L\psi^* = \begin{cases} \displaystyle\int_0^\eta \frac{S}{R}\,d\mu + a & \text{for } \theta < \beta \\[2ex] a & \text{for } \eta > \beta \end{cases}$$

We are now ready to verify the variational conditions. Since $R(\mu)$ is of one sign in $(0,\beta)$, (7.34) implies that $S(\mu)$ has a zero in $(0,\beta)$. According to (7.26), $S(\beta) = 0$; since S is a quadratic polynomial in μ^2, it can have no more than these two roots on the positive axis and furthermore these two roots are simple; we draw now some consequences:

Since the leading term in S is $12t\mu^4$, $S(\mu) > 0$ for μ large. It follows therefore that $S(\mu) > 0$ for $\mu > \beta$; then, since S has a simple

root at β, $S(\mu) < 0$ for $\beta < \mu$ and near β. Then, since S has exactly one simple root in $(0,\beta)$, $S(\mu) > 0$ for μ small, see Fig. 2:

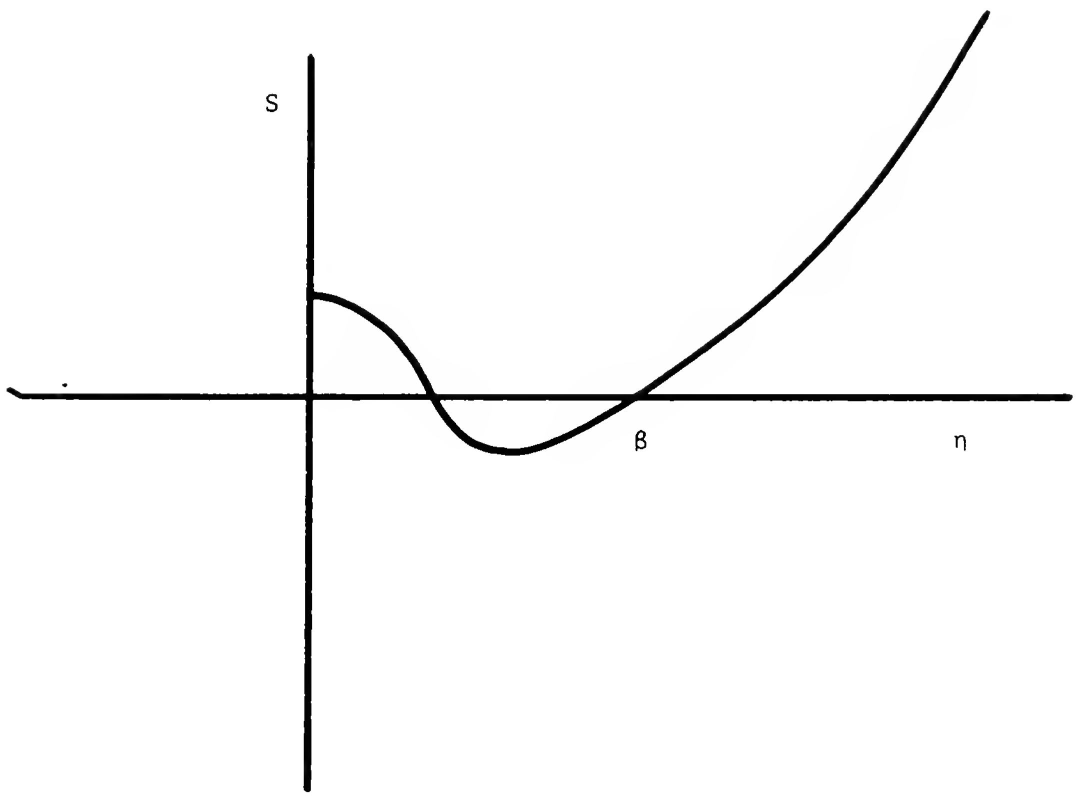

Figure 2

Since R is positive in $(0,\beta)$, it follows that for η small and $>$ 0,

$$(7.36) \qquad \int_0^\eta \frac{S(\mu)}{R(\mu)}\, d\mu > 0 .$$

We claim that (7.36) holds for all $\eta < \beta$; for clearly, it holds for $\eta < \eta_0$, where η_0 is the root of S in $(0,\beta)$. Since S is negative on (η_0,β), we deduce from (7.34) that for η between η_0 and β,

$$\int_0^\eta \frac{S}{R}\, d\mu > \int_0^\beta \frac{S}{R}\, d\mu = 0$$

This proves our contention.

Combining (733), (7.34) and (7.36) we have

$$a - L\psi^* \begin{cases} > 0 & \text{for } \eta < \beta \\ = 0 & \text{for } \beta < \eta \end{cases}$$

In view of (7.19)′ this shows that the variational conditions (7.15), (7.15)′ are satisfied.

We note finally that, since $S(\mu) > 0$ for $\mu > \beta$, it follows from (7.19)′ that $\psi^* \geqslant 0$ and therefore admissible.

To complete the proof of part (b), we have to show that as β varies from 0 to 1, x/t varies between -6 and 4. From (7.21) and (7.26) we can express

$$\frac{x}{t} = 12 \frac{\beta^4 + \beta_1\beta^2 + \gamma_2}{\beta^2 + \alpha} \tag{7.37}$$

The coefficients, γ_1 and γ_2 are given by formulas (7.22), (7.23) and (7.24) in terms of the complete elliptic integrals $I^{(j)}$ defined in (7.25). By elementary calculus we deduce

<u>Lemma 7.5</u>: As $\beta \to 0$

(i) $I^{(0)}(\beta) \simeq \pi/2$

(ii) $I^{(2)}(\beta) \simeq \frac{\pi}{4}\beta^2$

(iii) $I^{(4)}(\beta) \simeq \frac{3\pi}{16}\beta^4$

We deduce from this fact that as $\beta \to 0$

$$\alpha \simeq \frac{\beta^2}{2}, \quad \gamma_1 \simeq -\frac{1}{2}, \quad \gamma_2 \simeq \frac{\beta^2}{4}$$

Setting this into (7.37) gives

$$\frac{x}{t} \simeq 12 \frac{-\beta^2/4}{\beta^2/2} \sim -6 \quad \text{as } \beta \to) \tag{7.38}$$

Using intermediate calculus we deduce

<u>Lemma 7.6</u>: As $\beta \to 1$

(i) $I^0(\beta) \simeq \frac{1}{2}|\log(1-\beta)|$

(ii) $I^{(0)}(\beta) - I^{(2)}(\beta) \simeq 1$

(iii) $I^{(2)}(\beta) - I^{(4)}(\beta) \simeq 1/3$

We deduce from this that

$$\alpha \simeq -1 + I_0^{-1} \ , \quad \gamma_1 \simeq -1 \ , \quad \gamma_2 \simeq \frac{1}{3} I_0^{-1}$$

Setting this into (7.37) gives

$$(7.38)' \qquad \frac{x}{t} \simeq 12 \frac{(1/3)I_0^{-1}}{I_0^{-1}} = 4 \quad \text{as} \quad \beta \to 1$$

Since x/t as defined by (7.37) is a continuous function of β, it follows that as β varies from 0 to 1, x/t varies between -6 and 4. Since according to Theorem 3.12 for each x,t there can be at most one function ψ^* that satisfies the variational condition, it follows that the relation of β to x/t is one-to-one. This completes the proof of Theorem 7.4.

According to formulas (5.56), (5.57)

$$\overline{u} = -1 - \beta^2 - 2\alpha$$

$$\overline{u^2} = (1+\beta^2)^2 - \frac{4}{3}\beta^2 + \frac{4}{3}(1+\beta^2)\alpha$$

Setting the definitions (7.22)-(7.24) into these formulas yields

$$(7.39) \qquad \overline{u} = -1 - \beta^2 + 2I^{(2)}(\beta)/I^{(0)}(\beta)$$

and

$$(7.40) \qquad \overline{u^2} = 1 + \beta^4 + \frac{1}{3}\beta^2 - \frac{4}{3}(1+\beta^2)I^{(2)}(\beta)/I^{(0)}(\beta)$$

Theorem 7.7: (a) For $x < -6t$

$$\overline{u}(x,t) = 1 \ , \quad \overline{u^2}(x,t) = 1$$

(b) For $-6t < x < 4t$, $\overline{u}$ and $\overline{u^2}$ are given by (7.39) and (7.40), where β is a function of x/t defined by (7.37) and (7.22)-(7.24).

(c) for $4t < x$

$$\overline{u}(x,t) = 0 \ , \quad \overline{u^2}(x,t) = 0 \ .$$

Note that weak limits satisfy

$$0 < \overline{u}^2 \le \overline{u^2} \tag{7.41}$$

Using formulas (7.39), (7.40) we get amusing inequalities for complete elliptic integrals from (7.41).

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